

Research Paper

New Interpretation of Resonance

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Abstract: *In the theory of electric circuits, complex numbers are widely used along with real numbers to describe resonance processes in them, for example complex frequencies, complex resistances, complex voltages. However, their physical meaning is not explained. The present investigation presents the evidence that these concrete complex numbers and resonance on complex frequencies are physical reality.*

Keywords: electric circuits, resonance, real frequencies, complex frequencies, electromagnetic compatibility.

1. Introduction

Resonance, as a physical phenomenon, is well known in science and is widely used in engineering, including electric and communication engineering.

There is a comprehensive bibliography on the theory of electric circuits [1] – [8]. Nevertheless, there are still questions related to resonance, which have no answers in the scientific-technical literature, namely:

- Why several real resonance frequencies can be available in the same electric LCR-circuit of the second order?
- Why the real resonance frequencies do not coincide with real frequency of free oscillations in electric LCR-circuits of the second order?
- What is resonance on complex frequencies, what is its physical meaning?
- What are the complex frequencies and how to see them?
- What are the concrete complex numbers, what physical meaning do they have?

2. Complex numbers

After Cardano discovered the so-called imaginary unit $\sqrt{-1} = i$ back in 1545 (hereinafter the imaginary unit is referred to as j , as in electric engineering i denotes electric current), mathematics saw the appearance of complex numbers. And at present, they are already widely used. There is even a perfect theory of functions of a complex variable [9] – [11].

However, in terms of a human life experience, complex numbers remain completely incomprehensible until now, contrary to other numbers, positive and negative, zero, integer and fractional, rational and irrational.

They are so incomprehensible, that, for example, Newton did not include imaginary values into the concept of a number at all. And Leibniz said the following: “Imaginary numbers are a wonderful and marvelous shelter of a divine spirit, almost an amphibian of existence with non-existence” [12].

Over the recent years, the attitude to this problem has not changed. The physical meaning of complex numbers is still not explained in any textbook. Therefore, it is not clear how one has to understand, e. g., complex voltage and other complex physical values which are used in calculations of electric circuits. Since this complicated problem has not been solved yet, at least its existence should be recognized. This would allow at least thinking over this problem and discussing it.

Complex numbers in engineering sciences, such as theory of electric circuits, aerodynamics and hydrodynamics, optics, etc., are widely used. They are used so widely and efficiently that without using the concept of a complex number they simply cannot be imagined.

3. Complex frequencies

In the theory of electric circuits a symbolic method of calculation is widely used, in accordance with which resistance and conductivity of an electric capacitor and electric inductance coil are measured by imaginary numbers. Taking into account their imperfections which cause energy losses in them, resistance and conductivity of electric capacitor and electric inductance coil will be measured by complex numbers, contrary to resistor, resistance and conductivity of which are measured by real numbers. However, taking into account the presence of stray capacitance and stray inductance in a resistor, its resistance and conductivity will be also measured by complex numbers.

In a general case, mathematical expressions of complex resistances and complex conductivities of electric LCR-circuits always include a complex frequency which is given different interpretation by various authors. But these differences in interpretation of a complex frequency concept do not matter right now. It is possible to use any of these interpretations.

Another thing is important. Immittance functions of a complex frequency (which imply resistance, or conductivity, or transfer functions) of electric circuits containing different electric components, calculated this way, take zero or infinitely high absolute values at certain values of complex frequency. This happens because both the numerator and denominator of an immittance function are polynomials of a complex frequency. Therefore, by putting the polynomial to zero in the nominator and solving the respective algebraic equation, it is possible to find the values of the so-called “zeroes”, and by putting the polynomial to zero in the denominator and solving the respective algebraic equation, it is possible to find the values of so-called “poles”.

In the simplest case, when continuous sinusoidal oscillations constantly affect the electric circuit from the outside, the term “resonance frequency” is used instead of the above-mentioned “zeroes” and “poles” and processes taking place at these frequencies are referred to, respectively, as resonance processes.

Therefore, it would seem that “zeroes” and “poles” could also be referred to as resonance frequencies. And sometimes they are, indeed [13], [14]. But most specialists on the theory of electric circuits do not use the term “complex resonance frequency”, as the “zeroes” and “poles” are complex frequencies, unlike the above-mentioned resonance frequencies measured by real numbers. Therefore, if we allow the existence of the term “complex resonance frequency”, it would be necessary to recognize the existence of resonance at complex frequencies.

And it would be necessary to study this resonance at complex frequencies and to prove convincingly to physicists that it really exists.

This is what we will attempt to do.

4. Resonance at complex frequencies

As is known, in linear electric circuits resonance is understood as a phenomenon causing the following, as the real frequency of external impact approaches the real resonance frequency:

- amplitude of a forced component of response takes an extreme value;
- phase shift between external impact and a forced component of response becomes zero;
- frequency of free oscillations becomes equal to resonance frequency.

But it turns out that this resonance at real frequencies, having all of the above-mentioned characteristics, can take place only in electric LC-circuits which are not of any practical interest. In real electric LCR-circuits the above-mentioned manifestations of the resonance take place at

different real frequencies.

Let us consider two examples confirming the above statements.

The first example. Conductivity of the simplest electric LCR-circuit, given in Figure 1a, equals to:

$$Y(j\omega) = C \frac{\left(\frac{1}{LC} - \omega^2\right) + j\omega \frac{R}{L}}{\frac{R}{L} + j\omega} = C \frac{(\omega_0^2 - \omega^2) + j\omega 2\sigma_0}{2\sigma_0 + j\omega}$$

where $2\sigma_0 = R/L$; $\omega_0 = 1/\sqrt{LC}$.

Reactive component of this complex conductivity is

$$\text{Im}Y(j\omega) = C \frac{\omega(\omega^2 - \omega_0^2 + 4\sigma_0^2)}{\omega^2 + 4\sigma_0^2}$$

Putting it to zero, we obtain an equation, solving which we shall find the combination of two (but not one!) solutions, corresponding to one of features of resonance revealed during investigation of LC-circuits

$$\begin{cases} \omega'_{res1} = 0 \\ \omega''_{res1} = \sqrt{\omega_0^2 - 4\sigma_0^2} = \omega_0 \frac{\sqrt{Q^2 - 1}}{Q} \end{cases}$$

where $Q = \frac{\omega_0}{2\sigma_0} = \frac{1}{R} \sqrt{\frac{L}{C}} = \frac{\rho}{R}$ is the circuit Q-factor.

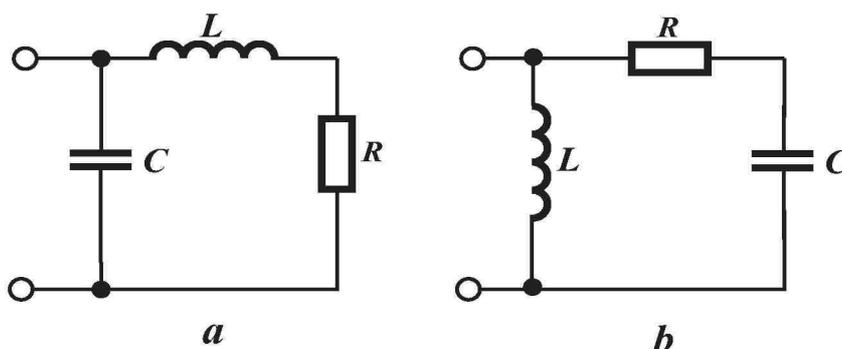


Fig.1. Electric LCR-circuits under consideration

Int. J. Pure Appl. Sci. Technol., 1(2) (2010), 1-12

Modulus of a complex conductivity of this oscillating LCR-circuit is:

$$|Y(j\omega)| = C \sqrt{\frac{(\omega^2 - \omega_0^2)^2 + 4\sigma_0^2 \omega^2}{\omega^2 + 4\sigma_0^2}}$$

As a result of investigation for extremum of its radicand, i.e. by solving the equation

$$\frac{d}{d\omega} \left[\frac{(\omega^2 - \omega_0^2)^2 + 4\sigma_0^2 \omega^2}{\omega^2 + 4\sigma_0^2} \right] = \frac{2\omega[\omega^4 + 8\sigma_0^2 \omega^2 - (\omega_0^4 + 8\sigma_0^2 \omega_0^2 - 16\sigma_0^4)]}{(\omega^2 + 4\sigma_0^2)^2} = 0$$

we obtain another combination of two (but not one!) solutions, corresponding to another feature of resonance revealed during studies of LC-circuits

$$\begin{cases} \omega'_{res2} = 0 \\ \omega''_{res2} = \sqrt{\omega_0 \sqrt{\omega_0^2 + 8\sigma_0^2} - 4\sigma_0^2} = \omega_0 \frac{\sqrt{Q\sqrt{Q^2 + 2} - 1}}{Q} \end{cases}$$

As is seen, finally we obtained four resonance frequencies for this LCR-circuit, moreover, $\omega'_{res1} = \omega'_{res2}$, and $\omega''_{res1} \neq \omega''_{res2}$.

The second example. Complex conductivity of LCR-circuit, given in Figure 1b, equals to

$$Y(j\omega) = \frac{1}{L} \frac{\left(\frac{1}{LC} - \omega^2\right) + j\omega \frac{1}{RC}}{j\omega \left(\frac{1}{LC} + j\omega \frac{1}{RC}\right)} = \frac{1}{L} \frac{(\omega_0^2 - \omega^2) + j\omega 2\sigma_0}{j\omega(\omega_0^2 + j2\sigma_0\omega)}$$

where $2\sigma_0 = 1/RC$; $\omega_0 = 1/\sqrt{LC}$.

Its reactive component

$$\text{Im}Y(j\omega) = \frac{1}{L} \frac{\omega^2(\omega_0^2 - 4\sigma_0^2) - \omega_0^4}{\omega(\omega_0^4 + 4\sigma_0^2 \omega^2)}$$

takes a zero value at the frequency corresponding to the first feature of the resonance

$$\omega_{res1} = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 4\sigma_0^2}} = \omega_0 \frac{Q}{\sqrt{Q^2 - 1}}$$

Int. J. Pure Appl. Sci. Technol., 1(2) (2010), 1-12

$$\text{where } Q = \frac{\omega_0}{2\sigma_0} = \frac{1}{R} \sqrt{\frac{L}{C}} = \frac{\rho}{R}.$$

Modulus of complex conductivity of the same LCR-circuit equals to:

$$|Y(j\omega)| = \frac{1}{L} \sqrt{\frac{(\omega_0^2 - \omega^2)^2 + 4\sigma_0^2 \omega^2}{\omega^2(\omega_0^4 + 4\sigma_0^2 \omega^2)}}$$

Studying its radicand for extremum, i.e., by solving the equation

$$\frac{d}{d\omega} \left[\frac{(\omega_0^2 - \omega^2)^2 + 4\sigma_0^2 \omega^2}{\omega^2(\omega_0^4 + 4\sigma_0^2 \omega^2)} \right] = \frac{2\omega[\omega^4(-\omega_0^4 - 8\omega_0^2 \sigma_0^2 + 16\sigma_0^4) + 8\omega^2 \sigma_0^2 \omega_0^4 + \omega_0^8]}{\omega_0^2 [(\omega_0^2 - \omega^2)^2 + 4\sigma_0^2 \omega^2]^2} = 0$$

we find a combination of resonance frequencies, corresponding to the second feature of the resonance

$$\left[\begin{array}{l} \omega'_{res2} = 0 \\ \omega_{res2} = \omega_0 \sqrt{\frac{\omega_0^3 \sqrt{\omega_0^2 + 8\sigma_0^2} + 4\sigma_0^2 \omega_0^2}{\omega_0^4 + 8\sigma_0^2 \omega_0^2 - 16\sigma_0^4}} = \omega_0 \sqrt{\frac{Q^3 \sqrt{Q^2 + 2} + Q^2}{Q^4 + 2Q^2 - 1}} \end{array} \right.$$

Thus, we obtained three different resonance frequencies for the last electric LCR-circuit.

Similar results were obtained in investigation of other electric LCR-circuits.

As is seen, even in the simplest LCR-circuits of the second order several resonance frequencies turned out to exist, moreover, even their number in different electric circuits is different.

As far as we know, this phenomenon has not yet been explained by anyone.

Furthermore, the frequency of free oscillations $\omega_{free} = \sqrt{\omega_0^2 - \sigma_0^2}$ in electric LCR-circuits, contrary to electric LC-circuits, always equals to none of resonance frequencies. In the middle of last century this unexplainable phenomenon was pointed to by academician Mandelstam [15], who developed the theory of the so-called generalized resonance. However, the theory of generalized resonance did not give answers to this and other questions above.

Based on the foregoing, the interpretation of resonance at real frequencies has to be recognized as not quite perfect.

Int. J. Pure Appl. Sci. Technol., 1(2) (2010), 1-12

It is evident that resonance at complex frequencies should be understood as a phenomenon, causing the following, as the complex frequency of external impact approaches the complex frequency of free oscillations:

- amplitude of a forced component of response takes an extreme value;
- phase shift between external impact and a forced component of response turns to zero;
- frequency of free oscillations becomes equal to resonance frequency.

And it turns out that all the above statements are strictly observed at complex frequencies in any linear electric circuits. Anybody can make sure this is true both theoretically and practically. It is not difficult to make sure of this theoretically, as resonance complex frequencies are understood as the above-mentioned “zeroes” and “poles” with all the relevant consequences.

Thus, for the electric circuit given in Figure 1a, the immittance function of complex frequency $p = -\sigma \pm j\omega$ has a form

$$Y(p) = C \frac{p^2 + p2\sigma_0 + \omega_o^2}{p + 2\sigma_0}$$

And for the electric circuit given in Figure 1b, the immittance function has a form

$$Y(p) = \frac{1}{L} \frac{p^2 + p2\sigma_0 + \omega_o^2}{p(p + 2\sigma_0)}$$

As is seen, the conductivity of both circuits indeed takes zero value at complex-conjugate resonance frequencies of effect equal to complex-conjugate frequencies of free oscillations $p_{res} = p_{free} = -\sigma_0 \pm j\sqrt{\omega_0^2 - \sigma_0^2}$.

It is not difficult to make sure that the above-mentioned non-contradictory resonance in electric LC-circuits at real frequencies is indeed only a particular case of resonance at complex frequencies.

The use of vector diagrams at complex frequencies allows explaining why several real resonance frequencies appear during investigation of resonance at real frequencies [16].

Finally, using the interpretation of resonance at complex frequencies makes it possible to explain why the frequency of free oscillations is not equal to any of resonance frequencies in investigation of resonance on real frequencies. The matter is that the parameters of free oscillations in electric LCR-circuits are always determined by roots of characteristic equation, i.e. they exist only at complex frequencies $p_{free} = -\sigma_0 \pm j\omega_{free}$. Consequently, it is the nature of the physical phenomenon under investigation that makes it impossible to ignore complex numbers.

Thus, the interpretation of resonance at complex frequencies is not contradictory, contrary to the interpretation of resonance at real frequencies. And, moreover, it allows explaining the contradictions in interpretation of resonance at real frequencies.

It is more difficult to make sure in the perfect interpretation of resonance at complex frequencies experimentally, as in order to do this it will be necessary to minimize in the studied electric circuit the effect of a free component of response impeding the observations.

This circumstance is worthy of a special explanation. Let us recollect in this connection the well-known but often forgotten circumstance that at resonance at real frequencies the separation of forced and free components of response occurs by itself in due time. Actually, the transition process, as the free component of response is also called, always fades with time in steady electric LCR-circuits. And the forced component of response to input continuous oscillations also represents continuous oscillations. Consequently, after a certain period of time which corresponds to the duration of the transition process, the observer always has a possibility to record only forced oscillations in the studied electric circuit.

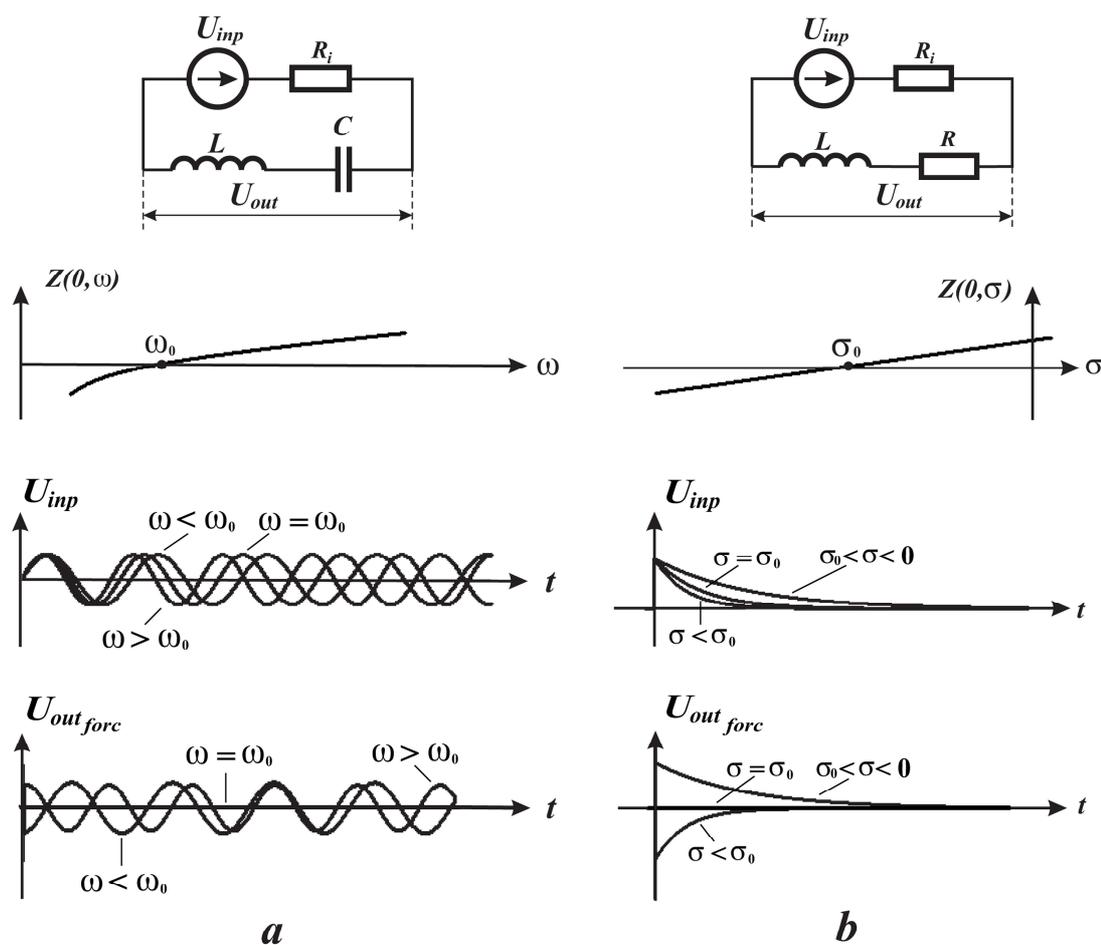


Fig. 2. Resonance in LC- and RL- two-pole circuits

However, in study of resonance at complex frequencies the impact on the electric circuit under consideration is made by exponential radio pulses or video pulses and is always limited in time. Therefore, for convenience of observation, this impact should be repeated. However, this is not all. It is also necessary to calculate parameters of the studied circuit to provide difference in duration (for example, by one order) between forced and free components of response. There are also other methods of solving this problem, which, however, are not considered here for simplicity.

In [16] – [18] the description of similar experiments is given. In particular, it is shown (see Figure 2a), that with change of a complex frequency $p = -\sigma$ of input exponential pulses U_{INP} (i.e. particular case of impact at complex frequencies) the polarity of the forced component of response $U_{OUT_{forc}}$ in transition through the resonance complex frequency $-\sigma_0$ changes its polarity similarly to the change of polarity by sinusoidal oscillations in transition of frequency of sinusoidal effect through the resonance frequency ω_0 (see Figure 2b). This result is far from evident, as any spectral component in electric RC- and RL-circuits is phase shifted not more than by $\pi/2_0$. However, when using amplitude-frequency characteristics at complex frequencies (see Figure 2a) it becomes quite clear.

The described experiments are rather simple and available to any electronic engineer. It is not difficult to check them.

It is also shown in [16] and [18] that the use of resonance at complex frequencies makes it possible to realize the filters of complex frequencies, which are more effective than the existing filters using resonance at real frequencies. Therefore, these filters will be useful in the solution of the problem of electromagnetic compatibility, as they will allow separating signals which are hardly separated by existing filters of real frequencies.

Some other practical problems with use of complex frequencies have been solved in invention [19].

Thus, the described investigations make it possible to argue that resonance at complex frequencies is confirmed experimentally and, consequently, it is real from the physical point of view.

5. Physical reality of complex frequencies

As follows from physical reality of resonance at complex frequencies the complex frequencies themselves are a physical reality. Consequently, their real and imaginary components are also real from the physical point of view, moreover, both of them influence the process of resonance in a similar way (see Figure 2).

Int. J. Pure Appl. Sci. Technol., 1(2) (2010), 1-12

Nevertheless, some doubts still remain after all the foregoing explanations and proofs, as well as the desire to obtain the answer to the question: how do we see the real and imaginary components of the complex frequency, see in the direct sense of the word, since a human thinks by visual images.

This is not an obligatory requirement. For example, no one can see the electric current, but we can easily feel it, if the voltage is high enough. It can be also measured by instruments, thus confirming to some extent the reality of this invisible physical value. Moreover, the electric and electronic devices are an integral part of life of almost all people.

However, technical devices using complex frequencies are not widely used yet. And this also supports the psychological distrust to the physical reality of complex frequencies.

So, it is very desirable to see a complex frequency in order to realize its existence. And it turns out to be possible.

Actually, it is easy to understand the meaning of real and imaginary components of complex frequency by taking a look at the oscillograph screen displaying fading harmonic oscillations.

Using the Euler's formula, it is possible to write down for them

$$e^{-\sigma t} \cos \omega t = [e^{(-\sigma + j\omega)t} + e^{(-\sigma - j\omega)t}] / 2$$

where t is the time;

$p = -\sigma \pm j\omega$ is the complex-conjugate frequencies of oscillations.

Consequently, the damping harmonic oscillations represent a semi-sum of exponents with complex exponents, which include complex-conjugate frequencies as a co-factor $p = -\sigma \pm j\omega$. Here, the real component $-\sigma$ of complex-conjugate frequencies of these oscillations characterizes the fading rate, and the conjugate imaginary components $\pm j\omega$ of complex frequency characterize the rate of polarity alternation of these oscillations. And both of these components are well seen.

6. Physical reality of other concrete complex numbers

From the physical reality of complex frequencies follows the physical reality of other complex physical values.

Indeed, as the inductive resistance equals to $X_L = p \times L$, i.e. it is a product of a real number and a complex number, both of which are real from the physical point of view, then the complex

Int. J. Pure Appl. Sci. Technol., 1(2) (2010), 1-12

inductive resistance is also physical reality.

Similar arguments easily prove that other concrete complex numbers, such as capacity susceptance $X_C = p \times C$, voltage drop $U_L = X_L \times I$ at inductance L , current passing through capacitor C $I_C = X_C \times U$ etc. are also physical reality.

Moreover, by studying in a similar way the resonance in oscillation systems of other physical nature (for example, mechanical), it is possible to come to the conclusion about physical reality of other concrete complex numbers, for example, distances.

However, it should be taken into consideration that any concrete complex numbers acquire their imaginary component only during an oscillation process under the impact of exponential video pulses or radio pulses. Otherwise they do not contain the imaginary component. And, obviously, special devices are necessary for recording this imaginary component, as, for example, for recording invisible magnetic field, ionizing radiation, electric current, etc.

4. Conclusions

Resonance processes in electric circuits take into account the value of complex (but not real) frequencies of input impact and the free component of response. This proves the physical reality of complex frequencies and resonance at complex frequencies.

As complex electric resistance, complex electric conductivity and other complex values used in calculation of electric circuits are the functions of complex frequency, they are also physical reality.

Taking into account the fact that resonance can take place not only in electric circuits, it is possible to make a conclusion that other concrete complex numbers describing the respective oscillation processes, are also physical reality.

But these physically real values are complex only for the time of damped oscillations. In other words, only for this time the described above additional dimensions (i.e. imaginary components of complex numbers) appear in our physical world [20].

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Int. J. Pure Appl. Sci. Technol., 1(2) (2010), 1-12

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