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Research Paper

On a New Class of Harmonic Univalent Functions Defined by Fox-Wright Generalized Hypergeometric Function

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Abstract: *Using the Fox-Wright generalization of the classical hypergeometric function, we introduce and study a new class of harmonic univalent functions in the open unit disc. We establish some results involving coefficient condition, distortion bounds, extreme points, convex combinations, convolution and neighborhoods. We also discuss a class preserving integral operator.*

Keywords: Harmonic functions, Fox-Wright generalized hypergeometric function, Integral operator.

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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that

$$|h'(z)| > |g'(z)|, z \in D.$$

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the open unit disc $U = \{z \in C : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. (1)$$

In 1984 Clunie and Sheil- Small [3] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds, (see also [4], [9], [10]).

Note that S_H reduces to the class S of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

A function of the form (1) is harmonic starlike of order $\alpha, (0 \le \alpha < 1)$ for $|z| = r < 1$ if

$$\frac{\partial}{\partial\theta}(\arg(fre^{i\theta})) = \Re\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\} > \alpha.$$

The class of all harmonic starlike functions of order α is denoted by S_α^* and extensively studied by Jahangiri [7]. The case $\alpha = 0$ and $\alpha = b_1 = 0$ were studied by Silverman and Silvia [13] and Silverman [12], (see also [1]). In [7] Jahangiri proved that the coefficient condition

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{k-\alpha}{1-\alpha} |b_k| \le 1 (2),$$

is sufficient condition for functions $f = h + \overline{g}$ to be harmonic starlike of order α. If we put $\alpha = 0$ in above inequalities then we obtain sufficient condition for function $f = h + \overline{g}$ belonging to the class S_H^* of harmonic starlike functions.

Further, we denote by V_H the subclass of S_H consisting of functions of the form $f = h + \overline{g}$, where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, g(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k, |b_1| < 1. (3)$$

The Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k (4)$$

and

$$\varphi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k, (5)$$

is defined by $(\phi * \varphi)(z) = \phi(z) * \varphi(z) = z + \sum_{k=2}^{\infty} \lambda_k \mu_k z^k$.

For positive real parameters $\alpha_1, A_1, ..., \alpha_p, A_p$ and $\beta_1, B_1, ..., \beta_q, B_q (p, q \in N_0 = N \cup \{0\})$ satisfying the condition that $1 + \sum_{k=1}^{q} B_k - \sum_{k=1}^{p} A_k \ge 0 (z \in U)$, the Fox-Wright generalization

$$ {}_p\psi_q[(\alpha_1, A_1),...,(\alpha_p, A_p);(\beta_1, B_1),...,(\beta_q, B_q);z] = {}_p\psi_q[(\alpha_k, A_k)_{1,p}(\beta_k, B_k)_{1,q};z] $$

of the hypergeometric function ${}_pF_q(\alpha_1,...,\alpha_p,\beta_1,...,\beta_q,z)$ is defined by [14]; (see also [11]).

$$ {}_p\psi_q[(\alpha_k, A_k)_{1,p}(\beta_k, B_k)_{1,q};z] = \sum_{k=0}^{\infty}\{\prod_{n=1}^{p}\Gamma(\alpha_n + k)A_n\}\{\prod_{n=1}^{q}\Gamma(\beta_n + kB_n)\}^{-1}\frac{z^k}{k!}(z \in U). $$

If $A_n = 1(n = 1,..., p)$ and $B_n = 1(n = 1,...,q)$, then we have the following obvious relationship:

$$ \Theta_p\psi_q[(\alpha_n,1)_{1,p}(\beta_n,1)_{1,q};z] = {}_pF_q(\alpha_1,...,\alpha_p,\beta_1,...,\beta_q,z) $$

$$ = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_p)_k}{(\beta_1)_k...(\beta_q)_k}\frac{z^k}{k!} (6) $$

where ${}_pF_q(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q,z)$ is the generalized hypergeometric function (see for details [4], $(\alpha_k = \alpha(\alpha+1)...(\alpha+k-1)$ is the familiar Pochhammer symbol, and Θ is given by

$$ \Theta = (\prod_{n=0}^{p}\Gamma(\alpha_n))^{-1}(\prod_{n=0}^{q}\Gamma(\beta_n))(7). $$

By using the generalized hypergeometric function, Dziok and Srivastava [6] introduced a linear operator which was subsequently extended by Dziok and Raina [5] by using the Fox-Wright generalized hypergeometric function.

Let $W[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}]: S \to S$ be a linear operator defined by

$$ W[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}]\phi(z) = \{\Theta_z \; {}_p\psi_q[(\alpha_n, A_n)_{1,p};(\beta_n, B_n)_{1,q};z]\} * \phi(z), $$

then on using (4) and (7), we get

$$ W[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}]\phi(z) = z + \sum_{k=2}^{\infty}\Theta\sigma_k(\alpha_1)\lambda_k z^k, (8) $$

where Θ is defined by (7), and $\sigma_k(\alpha_1)$ is given by

$$ \sigma_k(\alpha_1) = \frac{\Theta\Gamma(\alpha_1 + A_1(k-1))...\Gamma(\alpha_p + A_p(k-1))}{(k-1)!\Gamma(\beta_1 + B_1(k-1))...\Gamma(\beta_q + B_q(k-1))}. $$

For convenience sake, we adopt the contracted notation $W_q^p[\alpha_1]$ to represent the following:

$$ W_q^p[\alpha_1]\phi(z) = W[(\alpha_1, A_1),...,(\alpha_p, A_p);(\beta_1, B_1),...,(\beta_q, B_q)]\phi(z), (10) $$

which we use in the sequel throughout. The linear operator $W_q^p[\alpha_1]$ contains the Dziok-Srivastava operator (see [6]), and as its various special cases contain such linear operators as the Hohlov operator, Carlson-Shaffer operator, Ruscheweyh derivative operator, generalized Bernardi-Libera-Livingston operator and fractional derivative operator. Details and references about these operators can be found in [5] and [6].

In view of the relationship (6) and the linear operator (8) for the harmonic function $f = h + \overline{g}$ given by (1), we define the operator

$$W_q^p[\alpha_1]f(z) = W_q^p[\alpha_1]h(z) + \overline{W_q^p[\alpha_1]g(z)}, (11)$$

and introduce below a new subclass $\overline{L_H}([\alpha_1], \beta)$ of S_H in terms of the operator defined by (11).

Definition 1.1: For $1 < \beta \le 2, z \in U$ and $W_q^p[\alpha_1]f(z)$ is defined by (11), we let $L_H([\alpha_1], \beta)$ denote the subclass of S_H consisting of harmonic functions $f \in S_H$ of the form (1) that satisfy the condition

$$\Re\{\frac{W_q^p[\alpha_1]h(z) + (-1)^n \overline{W_q^p[\alpha_1]g(z)}}{z}\} < \beta. (12)$$

We further let $\overline{L_H}([\alpha_1], \beta) = L_H([\alpha_1], \beta) \cap V_H$.

We deem it appropriate to mention here some of the useful subclasses which stem from the class $L_H([\alpha_1], \beta)$ defined above by (12) reduces to the classes which we illustrate below.

(1) If we put $A_n = 1 (n = 1,..., p)$ and $B_n = 1(n = 1,..., q)$, then the family $L_H([\alpha_1], \beta)$ defined by (12) reduces to the class denoted by $HL_H([\alpha_1], \beta)$ which satisfies the inequality:

$$\Re\{\frac{H_q^p[\alpha_1]h(z) + (-1)^n \overline{H_q^p[\alpha_1]g(z)}}{z}\} < \beta,$$

where $H_q^p([\alpha_1])$ is the Dziok-Srivastava operator [6].

(2) Next, in view of the relationship $W_1^2([a,1;c] = L(a,c)f(z)$, we obtain a class $GL_H([\alpha_1], \beta)$ satisfying the inequality:

$$\Re\{\frac{L(a,c)h(z) + (-1)^n \overline{L(a,c)g(z)}}{z}\} < \beta,$$

where $L(a,c)$ is the Carlson-Shaffer operator [2].

(3) Also, by noting the relationship $W_1^2([\lambda + 1,1;1]) = D^\delta f(z)$, we arrive at the class $RL_H([\alpha_1, \beta)$ which satisfies the inequality:

$$\Re\{\frac{D^\delta h(z) + (-1)^n \overline{D^\delta g(z)}}{z}\} < \beta,$$

where $D^\delta f(z)(\delta > -1)$ is the Ruscheweyh derivative operator [11](also see [8]).

(4) Lastly, in view of the relationship $W_1^2([2,1;2-\mu]) = \Omega_z^\mu f(z)$, we obtain another class $FL_H([\alpha_1, \beta)$ satisfying the condition that

$$\Re\{\frac{\Omega_z^\mu[\alpha_1]h(z) + (-1)^n \overline{\Omega_z^\mu[\alpha_1]g(z)}}{z}\} < \beta,$$

where Ω_z^μ is the Srivastava-Owa fractional derivative operator [15] given by

$$\Omega_z^\mu f(z) = \Gamma(2-\mu) z^\mu D_z^\mu f(z) (0 \le \mu < 1).$$

In this paper, we study coefficient bounds,distortion bounds, extreme points, convex combinations, convolution condition, neighborhood problems and discuss a class preserving integral operator.

2. Main Results

In our first theorem, we introduce a sufficient coefficient for functions to be in $L_H([\alpha_1], \beta)$.

Theorem 2.1: Let $f(z) = h(z) + \overline{g(z)}$ be such that h and g given by (1). Furthermore, let

$$\sum_{k=2}^{\infty} \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \sigma_k(\alpha_1) |b_k| \le \beta - 1 (13)$$

Then f is sense-preserving, harmonic univalent in U and $f \in L_H([\alpha_1], \beta)$.

Proof: If $z_1 \ne z_2$, then

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \ge 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right|$$

$$= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|}$$

$$\ge 1 - \frac{\sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} |b_k|}{1 - \sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} |a_k|}$$

$$\ge 0.$$

Hence, $f(z)$ is univalent in U.
Now, $f(z)$ is sense-preserving in U. This is because

$$|h'(z)| \ge 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1}$$

$$> 1 - \sum_{k=2}^{\infty} k\,|a_k|$$

$$\geq 1 - \sum_{k=2}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} |a_k|$$

$$\geq \sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} |b_k|$$

$$\geq \sum_{k=1}^{\infty} k\,|b_k|$$

$$> \sum_{k=1}^{\infty} k\,|b_k|\,|z|^{k-1}$$

$$\geq |g'(z)|.$$

It remains to show that $f(z) \in L_H([\alpha_1], \beta)$. Using the fact that $\Re w < \beta$, if and only if, $|w-1| < |w+1-2\beta|$, it suffices to show that

$$\left| \frac{\dfrac{W_q^p[\alpha_1]h(z) + (-1)^n \overline{W_q^p[\alpha_1]g(z)}}{z} - 1}{\dfrac{W_q^p[\alpha_1]h(z) + (-1)^n \overline{W_q^p[\alpha_1]g(z)}}{z} - (2\beta - 1)} \right| < 1, z \in U$$

We have

$$\left| \frac{\dfrac{z + \sum_{k=2}^{\infty} \sigma_k(\alpha_1) a_k z^k + (-1)^n \sum_{k=2}^{\infty} \overline{\sigma_k(\alpha_1) b_k z^k}}{z} - 1}{\dfrac{z + \sum_{k=2}^{\infty} \sigma_k(\alpha_1) a_k z^k + (-1)^n \sum_{k=2}^{\infty} \overline{\sigma_k(\alpha_1) b_k z^k}}{z} - (2\beta - 1)} \right|$$

$$= \left| \frac{\sum_{k=2}^{\infty} \sigma_k(\alpha_1) a_k z^{k-1} + (-1)^n \dfrac{\bar{z}}{z} \sum_{k=1}^{\infty} \overline{\sigma_k(\alpha_1) b_k z^{k-1}}}{2(\beta - 1) - \sum_{k=2}^{\infty} \sigma_k(\alpha_1) a_k z^{k-1} + (-1)^n \dfrac{\bar{z}}{z} \sum_{k=1}^{\infty} \overline{\sigma_k(\alpha_1) b_k z^k}} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty} \sigma_k(\alpha_1) |a_k|\,|z|^{k-1} + \sum_{k=1}^{\infty} \sigma_k(\alpha_1) |b_k|\,|z|^{k-1}}{2(\beta - 1) - \sum_{k=2}^{\infty} \sigma_k(\alpha_1) |a_k|\,|z|^{k-1} - \sum_{k=1}^{\infty} \sigma_k(\alpha_1) |b_k|\,|z|^{k-1}}$$

$$\leq \frac{\sum_{k=2}^{\infty}\sigma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty}\sigma_k(\alpha_1)|b_k|}{2(\beta-1) - \sum_{k=2}^{\infty}\sigma_k(\alpha_1)|a_k| - \sum_{k=1}^{\infty}\sigma_k(\alpha_1)|b_k|} \leq 1, \qquad \text{by (12).}$$

Hence the proof is complete.
The harmonic univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty}\frac{\beta-1}{\sigma_k(\alpha_1)}x_k z^k + \sum_{k=1}^{\infty}\frac{\beta-1}{\sigma_k(\alpha_1)}\overline{y_k z^k},$$

where $1 < \beta \leq 2$ and $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$, show that the coefficient bound given by (13) is sharp.

In the following theorem we show that the condition (13) is also necessary for the function $f = h + \overline{g}$ where h and g are of the form (3).

Theorem 2.2: Let $f(z) = h(z) + \overline{g(z)}$ be defined by (3). Then $f_n \in L_H([\alpha_1], \beta)$ if and only if

$$\sum_{k=2}^{\infty}\sigma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty}\sigma_k(\alpha_1)|b_k| \leq \beta - 1 \quad (14)$$

Proof: Since $\overline{L_H}([\alpha_1], \beta) \subset L_H([\alpha_1], \beta)$, we only need to prove the "only if" part part of the theorem. For functions f_n of the form (3), we note that the condition

$$\Re\left\{\frac{W_q^p[\alpha_1]h(z) + (-1)^n\overline{W_q^p[\alpha_1]g(z)}}{z}\right\} < \beta$$

is equivalent to

$$\Re\left\{1 + \sum_{k=2}^{\infty}\sigma_k(\alpha_1)a_k z^{k-1} + (-1)^n\frac{\overline{z}}{z}\sum_{k=1}^{\infty}\overline{\sigma_k(\alpha_1)b_k z^{k-1}}\right\}$$

$$\leq 1 + \sum_{k=2}^{\infty}\sigma_k(\alpha_1)|a_k||z|^{k-1} + (-1)^n\sum_{k=1}^{\infty}\sigma_k(\alpha_1)|b_k||z|^{k-1}$$

$$< \beta.$$

The above required condition must hold for all values of $z \in U$. Upon choosing the values of z on the real axis and let $z \to 1^-$, we have

$$\sum_{k=2}^{\infty}\sigma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty}\sigma_k(\alpha_1)|b_k| \leq \beta - 1,$$

which is the required condition.

Sharpness of (14) can be seen by the harmonic univalent function

$$f_n(z) = z + \sum_{k=2}^{\infty}\frac{\beta-1}{\sigma_k(\alpha_1)}x_k z^k + (-1)^n\sum_{k=1}^{\infty}\frac{\beta-1}{\sigma_k(\alpha_1)}y_k \overline{z}^{k},$$

where $1<\beta\le 2, x_k\ge 0, y_k\ge 0$ and $\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k\le 1$ belongs to the class $\overline{L_H}([\alpha_1],\beta)$.
The following theorem gives the distortion bounds for functions in $\overline{L_H}([\alpha_1],\beta)$.

Theorem 2.3: Let $f\in\overline{L_H}([\alpha_1],\beta)$. Then for $|z|=r<1$ we have

$$|f(z)|\le(1+|b_1|)r+\frac{(\beta-1-|b_1|)}{\sigma_2(\alpha_1)}r^2, (15)$$

and

$$|f(z)|\ge(1-|b_1|)r-\frac{(\beta-1-|b_1|)}{\sigma_2(\alpha_1)}r^2, (16)$$

Proof: Let $f\in\overline{L_H}([\alpha_1],\beta)$. Then

$$|f(z)|=|z-\sum_{k=2}^{\infty}a_kz^k+(-1)^n\sum_{k=1}^{\infty}b_k\overline{z^k}|$$

$$|f(z)|\le r+\sum_{k=2}^{\infty}|a_k|r^k+\sum_{k=1}^{\infty}|b_k|r^k$$

$$=r+|b_1|r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k$$

$$\le(r+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2.$$

By Theorem 2.2, we can obtain:

$$\le(1+|b_1|)r+\frac{1}{\sigma_2(\alpha_1)}\sum_{k=2}^{\infty}\sigma_2(\alpha_1)(|a_k|+|b_k|)]r^2$$

$$\le(1+|b_1|)r+\frac{1}{\sigma_2(\alpha_1)}\sum_{k=2}^{\infty}\sigma_k(\alpha_1)(|a_k|+|b_k|)]r^2$$

$$\le(1+|b_1|)r+\frac{(\beta-1-|b_1|)}{\sigma_2(\alpha_1)}r^2,$$

and

$$|f(z)|\ge(1-|b_1|)r-\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k$$

$$\ge(1-|b_1|)r-\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$

$$\ge(1-|b_1|)r-\frac{1}{\sigma_2(\alpha_1)}\sum_{k=2}^{\infty}\sigma_2(\alpha_1)(|a_k|+|b_k|)r^2$$

$$\geq (1-|b_1|)r - \frac{1}{\sigma_2(\alpha_1)}\sum_{k=2}^{\infty}\sigma_k(\alpha_1)(|a_k|+|b_k|)r^2$$

$$\geq (1-|b_1|)r - \frac{(\beta-1-|b_1|)}{\sigma_2(\alpha_1)}r^2.$$

Hence the proof is complete.
The following covering result follows from inequality (16).

Corollary 2.1: Let $f \in \overline{L_H}([\alpha_1],\beta)$. Then

$$\{w:|w|< \frac{\sigma_2(\alpha_1)-\beta+1}{\sigma_2(\alpha_1)} + (1-\sigma_2(\alpha_1))|b_1|\} \subset f(U).$$

Next we determine the extreme points of closed convex hulls of $\overline{L_H}([\alpha_1],\beta)$ denoted by $clco\overline{L_H}([\alpha_1],\beta)$.

Theorem 2.4: Let f be given by (3). Then $f \in clco\overline{L_H}([\alpha_1],\beta)$ if and only if

$$f(z) = \sum_{k=1}^{\infty}(\lambda_k h_k(z) + \mu_k g_k(z)),$$

where $h_1(z) = z$

$$h_k(z) = z + \frac{\beta-1}{\sigma_k(\alpha_1)}z^k, k = 2,3,...$$

and

$$g_k(z) = z + (-1)^n \frac{\beta-1}{\sigma_k(\alpha_1)}\overline{z}^k, k = 1,2,...$$

where $\lambda_k \geq 0, \mu_k \geq 0$ and $\sum_{k=1}^{\infty}(\lambda_k + \mu_k) = 1$.

In particular, the extreme points of $\overline{L_H}([\alpha_1],\beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof: Suppose that

$$f(z) = \sum_{k=1}^{\infty}(\lambda_k h_k(z) + \mu_k g_k(z))$$

$$= z + \sum_{k=2}^{\infty}(\frac{\beta-1}{\sigma_k(\alpha_1)})\lambda_k z^k + (-1)^n\sum_{k=1}^{\infty}(\frac{\beta-1}{\sigma_k(\alpha_1)})\mu_k \overline{z}^k.$$

Then

$$\sum_{k=2}^{\infty}(\frac{\sigma_k(\alpha_1)}{\beta-1})(\frac{\beta-1}{\sigma_k(\alpha_1)}\lambda_k) + \sum_{k=1}^{\infty}(\frac{\sigma_k(\alpha_1)}{\beta-1})(\frac{\beta-1}{\sigma_k(\alpha_1)}\mu_k)$$

$$= \sum_{k=2}^{\infty}\lambda_k + \sum_{k=1}^{\infty}\mu_k) = 1-\lambda_1 \leq 1.$$

Thus $f \in clco\overline{L_H}([\alpha_1],\beta)$.

Conversely, suppose that $f \in clco\overline{L_H}([\alpha_1], \beta)$.
Letting

$$\lambda_k = \frac{\sigma_k(\alpha_1)}{\beta - 1} | a_k |, k = 2,3,...$$

$$\mu_k = \frac{\sigma_k(\alpha_1)}{\beta - 1} | b_k |, k = 1,2,3,...$$

Also

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k .$$

Hence, we obtain

$$f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k(z) + \mu_k g_k(z)).$$

This completes the proof of the theorem.

Theorem 2.5: $\overline{L_H}([\alpha_1], \beta) \subseteq S_H^*$ where $1 < \beta \leq 2$.

Proof: If $f \in \overline{L_H}([\alpha_1], \beta)$, then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | a_k | + \sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | b_k | \leq 1.$$

But

$$\sum_{k=2}^{\infty} k | a_k | + \sum_{k=1}^{\infty} k | b_k |$$

$$\sum_{k=2}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | a_k | + \sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | b_k |$$

$$\leq 1.$$

Therefore $f \in S_H^*$.

Theorem 2.6: The class $\overline{L_H}([\alpha_1], \beta)$ is closed under convex combination.

Proof: Suppose that $f_i \in \overline{L_H}([\alpha_1], \beta)$, for $i = 1,2,...$, where

$$f_i(z) = z + \sum_{k=2}^{\infty} | a_{k_i} | z^k + (-1)^n \sum_{k=1}^{\infty} | b_{k_i} | \overline{z}^k .$$

To prove the theorem it is enough to show that

$$\sum_{i=1}^{\infty} \lambda_i f_i \in \overline{L_H}([\alpha_1], \beta),$$

where $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{\infty} \lambda_i = 1$.

Now, by Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|a_{k_i}|+\sum_{k=1}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|b_{k_i}|\le 1.$$

Then

$$\sum_{i=1}^{\infty}\lambda_i f_i(z)=z+\sum_{k=2}^{\infty}(\sum_{i=1}^{\infty}\lambda_i|a_{k_i}|)z^k+\sum_{k=1}^{\infty}(\sum_{i=1}^{\infty}\lambda_i|b_{k_i}|)\overline{z}^{-k}.$$

By Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}(\sum_{i=1}^{\infty}\lambda_i|a_{k_i}|)+\sum_{k=1}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}(\sum_{i=1}^{\infty}\lambda_i|b_{k_i}|)$$

$$=\sum_{i=1}^{\infty}\lambda_i(\sum_{k=2}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|a_{k_i}|+\sum_{k=1}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|b_{k_i}|)$$

$$\le\sum_{i=1}^{\infty}\lambda_i=1.$$

Therefore

$$\sum_{i=1}^{\infty}\lambda_i f_i\in\overline{L_H}([\alpha_1],\beta).$$

This completes the proof of the theorem.
Now we show that the class $\overline{L_H}([\alpha_1],\beta)$ is closed under convolution.

Theorem 2.7: For $1<\beta\le\gamma\le 2$, let $f\in\overline{L_H}([\alpha_1],\beta)$ and $F\in\overline{L_H}([\alpha_1],\gamma)$.

Then $(f*F)(z)\in\overline{L_H}([\alpha_1],\beta)\subset\overline{L_H}([\alpha_1],\gamma)$.

Proof: Let $f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^{-k}\in\overline{L_H}([\alpha_1],\beta)$,

and

$$F(z)=z+\sum_{k=2}^{\infty}|A_k|z^k+(-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z}^{-k}\in\overline{L_H}([\alpha_1],\gamma).$$

Then

$$(f*F)(z)=z+\sum_{k=2}^{\infty}|a_kA_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_kB_k|\overline{z}^{-k}.$$

Since $F\in\overline{L_H}([\alpha_1],\gamma)$, we note that $|A_k|\le 1$ and $|B_k|\le 1$.

Now by Theorem 2.2, we get:

$$\sum_{k=2}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|a_kA_k|+\sum_{k=1}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|b_kB_k|$$

$$\le\sum_{k=2}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{\sigma_k(\alpha_1)}{\beta-1}|b_k|$$

$$\le 1.$$

Therefore $(f * F)(z) \in \overline{L_H}([\alpha_1], \beta) \subset \overline{L_H}([\alpha_1], \gamma)$.

Hence the proof is complete.

The δ - neighborhood of f is the set

$$N_\delta(f) = \{F : F(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \overline{z}^{-k}$$

and $$\sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \le \delta\}.$$

Theorem 2.8: Let $f \in \overline{L_H}([\alpha_1], \beta)$. If $F \in N_\delta(f)$, then F is harmonic starlike function, where $\delta \le 2 - \beta$.

Proof: Suppose that $F \in N_\delta(f)$.

Then

$$F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| \overline{z}^{-k},$$

and

$$\sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \le \delta.$$

Thus

$$\sum_{k=2}^{\infty} k|A_k| + \sum_{k=1}^{\infty} k|B_k|$$

$$\le \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + \sum_{k=2}^{\infty} k(|a_k + b_k|) + |b_1 - B_1| + |b_1|$$

$$\le \delta + \beta - 1$$

$$\le 1.$$

Therefore $F(z)$ is harmonic starlike function.

3. A Family of Class Preserving Integral Operator

We examine the closure properties of the class $\overline{L_H}([\alpha_1], \beta)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, (c > -1).$$

Theorem 3.1: Let $f(z) \in \overline{L_H}([\alpha_1], \beta)$ be of the form (3). Then $L_c(f(z)) \in \overline{L_H}([\alpha_1], \beta)$.

Proof: Let $f(z) \in \overline{L_H}([\alpha_1], \beta)$ be of the form (3). Then

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \overline{z}^k .$$

Now

$$L_c(f(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt}, (c > -1)$$

$$= \frac{c+1}{z^c}(\int_0^z t^{c-1}(t + \sum_{k=2}^{\infty} || a_k | t^k)dt + (-1)^n \overline{\int_0^z t^{c-1}(\sum_{k=1}^{\infty} | b_k | t^k)dt)}$$

$$= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} | a_k | z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k} | b_k | \overline{z}^k .$$

By Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | a_k | + \sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | b_k | \leq 1.$$

Therefore

$$\sum_{k=2}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} (\frac{c+1}{c+k} | a_k |) + \sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} (\frac{c+1}{c+k} | b_k |)$$

$$\leq \sum_{k=2}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | a_k | + \sum_{k=1}^{\infty} \frac{\sigma_k(\alpha_1)}{\beta - 1} | b_k |$$

$$\leq 1.$$

Hence $L_c(f(z)) \in \overline{L_H}([\alpha_1], \beta)$.

This completes the proof of the theorem.

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