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Research Paper

A Class of Generalized best Approximation Problems in a Fuzzy Normed Linear Space

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Abstract: *In this paper, a class of generalised best approximation problems is formulated in a fuzzy normed linear space.*

Keywords: Best approximation, fuzzy normed linear-space, semi-reflexive, proximal set, continuous operators.

Introduction:

The best approximation problem in normed linear spaces were considered by several authors including Burbu [1], singer [2]. Also in locally convex spaces some results were obtained by singer [3], Samanta [4]. The principal objective of this paper is to generalise the idea of best approximation problem in a fuzzy normed linear space setting and apply to optimal control.

Some Fundamental Definitions

Definition 1: Let X be any non empty set and $F(X)$ be the set of all Fuzzy sets on X .For $U, V \in F(X)$ and $k \in K$ the field of real numbers, define $U+V=\{(x + y, \lambda \wedge \mu)|(x, \lambda) \in U, (y, \mu) \in V\}$ and $kU =\{(k x , \lambda)|(x, \lambda) \in U\}$.

Definition 2: Let X be a linear space over K (field of real or complex numbers) then a fuzzy linear space $\widetilde{X} = X \times (0, 1]$ over the field K where the addition on $\widetilde{X}$ are defined by $(x , \lambda)+(y, \mu) = (x + y, \lambda \wedge \mu)$ $k(x ,\lambda) = (k x ,\lambda)$ is a fuzzy normed linear space if to every $(x , \lambda) \in \widetilde{X}$ there is corresponds a non-negative real number $\|(x ,\lambda)\|$,called fuzzy normed of (x, λ) in such a way that

(1) $\|(x , \lambda)\|=0$ iff $x = 0$ the zero element of X , $\lambda \in (0, 1]$,

(2) $\| k(x, \lambda)\| = | k | \|(x, \lambda)\|$ for $(x, \lambda)\in \widetilde{X}$ and all $k\in K$,

(3) $\|(x, \lambda)+(y, \mu)\|\leq\|(x , \lambda \wedge\mu)\|+\|(y , \lambda \wedge \mu)\|$ for all (x, λ), $(y, \mu)\in \widetilde{X}$,

(4) $\|(x, \vee_t \lambda_t)\| = \wedge_t \|(x, \lambda_t)\|$ for $\lambda_t \in (0,1]$.

Definition 3: Let X be a linear space over the field of real or complex numbers say K, then a fuzzy subset N of X x R (R is the set of real numbers) is called a fuzzy norm on X if and only if for all x $\in$ X and c $\in$ K,

(N_1) for all $t\in R$ with $t<0$, $N(x, t)=0$,

(N_2) for all $t\in R$ with $t>0$, $N(x, t) =1$ if and only if $x = 0$.

(N_3) for all $t\in$ R with $t>0$, $N(cx, t) = N(x ,t/|c|)$, if $c \neq 0$.

(N_4) for all $s, t\in R, x, u \in X$, $N(x+u, s + t) \geq \min\{N(x, s), N(u, t)\}$

(N_5) $N(x, \cdot)$ is a non-decreasing function of R and lim $N(x, t) = 1$,then the pair (x, N) is called fuzzy normed linear space.

Definition 4: Let (X, N) be a fuzzy normed linear space .we define $\| x \|_\alpha = \inf \{t: N(x, t) \geq \alpha\}, \alpha \in (0, 1)$. Then $\{\| \cdot \|_\alpha: \alpha \in (0, 1)\}$ is an ascending family of norms on X (or) α-norms on X corresponding to the fuzzy norm on X.

Definition 5: Let X be a non-empty set and *F(X)* be the set of all fuzzy sets in X. If $f\in F(X)$, then $f = \{(x, \mu)/x \in X$ and $\mu\in (0, 1]\}$. Clearly f is a bounded function for $|F(X)| \leq 1$. Let K be the space of real numbers then *F(X)* is a linear space over the field K where the addition and scalar multiplication are defined by $f + g = \{ (x, \mu)+(y, ŋ) \}=\{(x + y ,\mu \wedge ŋ) / (x, \mu)\in f$ and $(y, ŋ) \in g\}$

And $kf = \{(kf, \mu) / (x, \mu)\in f\}$, where $k\in K$.

The linear space *F(X)* is said to be normed space if for every $f\in F(X)$, there is associated a non-negative real number $\| f\|$ called the norm of f in such a way that

(1) $\| f \| = 0$ if and only if $f = 0$. For $\| f \| = 0$ if and only if $\{\|(x, \mu)\in f \} = 0$, $x = 0$, $\mu \in (0,1]$ if and only if $f = 0$.

(2) $\| kf \| =| k |\| f \|$, $k\in K$. For $\| k f \| = \{\| k(x, \mu) \| / (x, \mu) f, k\in K\} = \{| k |\|x, \mu \| / (x, \mu) \in f \} = |k| \| f \|$.

(3) $\| f + g \| \leq \| f \|+ \| g \|$ for every $f, g\in F(X)$.

For $\| f + g\| =\{\|(x, \mu) + (y, ŋ) \| / x , y \in X , \mu, ŋ\in (0,1]\}$

$= \{\| (x +y), (\mu \;\; ŋ) \| / x, y \in X, \mu, ŋ \in (0,1]\}\leq \{\| (x, \mu \wedge ŋ)\| + \| (y, \mu\wedge ŋ)\| / (x, \mu)\in f$ and $(y ,ŋ) \in g\} = \| f \| + \| g \|$.

Then $(F(X), \| \cdot \|)$ is a normed linear space.

Definition 6: A 2-fuzzy set on X is a fuzzy set on *F(X)*.

Definition 7: Let *F(X)* be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R$ (R, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on *F(X)*) if and only if,

(N_1) for all $t\in$ R with $t \leq 0$, $N (f_1, f_2, t) = 0$,

(N_2) for all $t\in R$ with $t \geq 0$, $N(f_1, f_2, t) = 1$, if and only if f_1 and f_2 are linearly dependent,

(N_3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1 and f_2,

(N_4) for all $t\in R$, with $t\geq 0$, $N(f_1, cf_2, t) = N(f_1, f_2, t / |c|)$ if $c \neq 0$, $c\in$ K (field),

(N_5) for all $s, t\in R$, $N(f_1, f_2+f_3, s + t) \geq \min \{N(f_1, f_2, s), N(f_1, f_3, t)\}$,

(N_6) $N(f_1, f_2,): (0, \infty) \rightarrow [0, 1]$ is continuous,
(N_7) $\lim_{t\to\infty} N(f_1, f_2, t) = 1$.

Then $(F(X), N)$ is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Let X be a linear space. Then $(\tilde{X}, \|.\|)$ is a fuzzy normed linear space. Let $\tilde{X}^*$ denote the class of all continuous convex function defined on X.

Definition 8: Let $F(X)$ be the set of all maps from X to I, where I = [0, 1]. Then λ is a fuzzy point in X, x its support and α its value, λ may be denoted by λ_x^α.

Let $\gamma \in F(X)$, then $\lambda_{xo}^{\alpha o} \in \gamma$ iff $\gamma(x_0) > \alpha_0$, [9].

Fuzzy point $\lambda_x^\alpha, \lambda_{x'}^{\alpha'}$ will be denoted by $\lambda, \lambda_i, i = 0,1,2,.....\lambda$ respectively.

Let X be a reference set, I= [0, 1] and let F (α) be the set of all maps from X to I. The partial order in

$F(X)$ is defined as follows $\gamma < \mu$

where $\gamma, \mu \in F(X)$ if and only if $\gamma(x) \leq \mu(x)$ *for all* $x \in X$.

Then $(F(X), <)$ is a complete lattice. The supremum and infimum are denoted by $\vee, \wedge$ respectively.

Definition 9:

(1) μ *is a fuzzy set in* $X \Leftrightarrow \mu \in F(X)$,
(2) $\mu_1 < \mu_2 \Rightarrow \mu_1(x) \leq \mu_2(x), \forall\ x \in X$.
(3) $(\vee_\alpha \mu_\alpha)(x) = \sup_\alpha \{\mu_\alpha(x)\}, \forall\ x \in X$.
(4) $(\wedge_\alpha \mu_\alpha)(x) = \inf_\alpha \{\mu_\alpha(x)\}, \forall\ x \in X$.
(5) $\mu^c(x) = 1 - \mu(x), \forall\ x \in X$.
(6) $\tilde{1}(x) = 1, \tilde{0}(x) = 0, \forall\ x \in X$.

Definition 10: Let $f : X \rightarrow Y, \mu \in F(X), \gamma \in F(Y)$

$$f(\mu)(y) = \sup_{x\in f^{-1}(y)} \{\mu(x)\},\ f(y) \neq \phi$$
$$= 0, \qquad f^{-1}(y) = \phi$$
$$f^{-1}(\gamma)(x) = \gamma(f(x)), \forall\ x \in X.$$

Definition 11: $S \subset F(X)$ satisfies the following conditions:

(1) $\tilde{I} \in S, \tilde{O} \in S.$

(2) $\mu \in S, \gamma \in S \Rightarrow \mu \wedge \gamma \in S$.

(3) $\mu_\alpha \in S \Rightarrow \vee_\alpha \mu_\alpha \in S$.

Then S is called a fuzzy topology on X and (X,S) a fuzzy topological space.

Definition 12: A fuzzy set $\lambda \in F(X)$ *such that* $\alpha \in (0,1)$ and

$$\lambda(x^1) = \alpha, \text{ if } x^1 = x$$
$$\lambda(x^1) = 0, \text{ if } x^1 \neq x$$

is called a fuzzy point in X (written λ), x *its* sup *port and* α *its value*, λ *may be denoted by* λ_x^α.

Definition 13: $\lambda_{x_0}^{\alpha_0}$ *is said to belong to* γ (*written* $\lambda_{x_0}^{\alpha_0} \in \gamma$) *if and only if* $\gamma(x_0) > \alpha_0$.

Definition 14[9]: Let $\mu \in F(X)$. Then μ is called a neighbourhood of p if $\exists \nu \in S$ such that $p \in \nu < \mu$ where $S \subset F(X)$.

Let N_p denote the system of all neighbourhoods of p. If $\mu \in F(x)$, *then* $\mu = \bigvee_{x \in X} \lambda_x$ *where* λ_x is a fuzzy point.From now on fuzzy point $\lambda_x^\alpha, \lambda_{x'}^{\alpha'}, i = 1, 2, ..., \lambda_{x'}^{\alpha'}$, will be denoted by $\lambda, \lambda_{i,} i = 0,1, ...\lambda$ respectively.

Definition 15[11]: A fuzzy linear space $\tilde{X} = X \times (0,1]$ over the number field K where the addition and scalar multiplication on $\tilde{X}$ are defined by

$$(x, \lambda) + (y, \mu) = (x + y, \lambda \wedge \mu),$$
$$\kappa(x, y) = (\kappa x, \lambda)$$

Then $\tilde{X}$ is a fuzzy linear space if to every $(x, \lambda) \in \tilde{X}$ there corresponds a non-negative real number, ||(x, λ)||, called the fuzzy normed of (x, λ) , such that

(1) $\| (x, \lambda) \| = 0$ *iff* $x = 0$ *the zero element of* $X, \lambda \in (0,1]$,

(2) $\| \kappa(x, \lambda) \| = | \kappa | \| (x, \lambda) \|, \forall (x, \lambda) \in \tilde{X}$ *and all* $\kappa \in K$,

(3) $\| (x, \lambda) + (y, \mu) \| \leq \| (x, \lambda \wedge \mu) \| + \| (y, \lambda \wedge \mu) \|, \forall (x, \lambda), (y, \mu) \in \tilde{X}$,

(4) $\| (x, \vee_t \lambda_t) \| = \wedge_t \| (x, \lambda_t) \|$ *for* $\lambda_t \in (0,1]$.

Then $\tilde{X}$ will be defined as fuzzy normed linear space.

Definition 16[11]: Let *X* be a any non-empty set and *F(X)* be the set of all fuzzy sets on X . For *U*, *V* ∈ *F(X)* and k Є*K* the field of real numbers, define

$$U,V = \{(x = y, \lambda \wedge \mu) \mid (x,\lambda) \in U, (y,\mu) \in V\},$$
$$and\ \kappa U = \{(\kappa x, \lambda) \mid (x,\lambda) \in U\}$$

Definition 17[11]: The fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \vee\{\mu_1(x) + \mu_2(x) : x = x_1 + x_2\}.$$

And for a scalar t of K and a fuzzy subset $\mu\, of\ X$, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(x/t) & if\ t \neq 0 \\ 0 \quad if\ t = 0\ and\ x \neq 0 \\ \vee\{\mu(y) \mid y \in X\} & if\ t = 0\ and\ x = 0 \end{cases}$$

Definition 18[11]: $\mu \in I^X$ is said to be

(1) convex if $tf + (1-t)f \subseteq f\ \ for\ each\ t \in (0,1]$,
(2) balanced if $tf \subseteq f\ \ for\ each\ t \in K\ with\ |t| \leq 1$,
(3) absorbing if $\vee\{tf(x) \mid t > 0\} = 1$ for all x ∈ X.

Statement of the Problem

Let E be a convex subset of $\tilde{X}$ and $\tilde{X}^*$ be the fuzzy conjugate space of all continuous linear functionals defined on $\tilde{X}$ consider the problem

$$\min_{m \in \tilde{X}} \{\frac{1}{2} \| m - x \|^2 + I_E(m)\}$$

where x is the given element of $\tilde{X}$ and I_E is the indicator function such that

$$I_E(m) = \begin{cases} 0 & if\ m \in E \\ +\infty & if\ m \notin E \end{cases}$$

To solve this problem let us consider the following definitions and theorem.

Definition 19: An element l$\in E$ is called a best approximation to x$\in \tilde{X}$ from E if $\|x - l\| \leq \|x - m\|,\ for\ all\ x \in E.$ (1)

Is the best approximation to x from E if and only if there exists $x_0^* \in \tilde{X}^*$ subject to $\|l\| + \|x_0^*\| \leq \|m\|, \forall m \in E$

$$\Rightarrow f(l) + f^*(x_0^*) \leq (x_0^*, m),\ \forall\ m \in E \tag{2}$$

Where $f(m) = \frac{1}{2} \sup_{f \in x^*} |(m - x\ ,\ f)|^2, m \in X, where\ f(m) = \frac{1}{2}\|m - x\|^2, m \in \tilde{X}.$

Theorem 1: *An element* $l \in E$ *is the best approximation to* $x \in \tilde{X}$ *from element of the convex set E if and only if there exists* $x_0^* \in \tilde{X}^*$ *such that*

$(i)\ \|x_0^*\| = \|l - x\|$

$(ii)\ (x_0^*, m - x) \geq \|l - x\|, \forall m \in E$

Proof: We have

$$f^*(x_0^*) = \sup_{m \in \tilde{X}}\{(x_0^*, m) - \frac{1}{2}\|m - x\|\} = (x_0^*, x) + \sup_{m \in \tilde{X}}\{(x_0^*, m) - \frac{1}{2}\|m\|^2\} = (x_0^*, x) + \frac{1}{2}\|x_0^*\|^2$$

and the optimality condition (2) becomes

$$\frac{1}{2}\|l - x\|^2 + \frac{1}{2}\|x_0^*\| \leq (x_0^*, m - x), \forall m \in E \qquad (3)$$

In particular,for m=l,we obtain $(\|l - x\| - \|x_0^*\|)^2 \leq 0.$

which implies condition(1).Consequently from inequality (3) condition (ii) follows.

Conversely, it is clear that condition (1) and (2) imply that l is a best approximation, because we have

$$\|l - x\|^2 \leq (x_0^*, m - n)$$
$$\leq \|x_0^*\|.\|m - n\|$$
$$\leq \|l - x\|.\|m - x\|, \forall m \in E$$

and so, we must have (1)

Corollary 1: *If* $l \in E$ *is a best approximation of* $x \in \tilde{X}$ *by elements of the convex set E, then the following minimax relation*

$$\|x - l\| = \min_{m \in E} \max_{\|x^*\|=1} (x^*, m - x)$$

$$= \max_{\|x^*\|=1} \min_{m \in E} (x^*, m - x) \qquad (4)$$

holds where $x^* \in X^*$

Proof: This follows clearly if we use the relationship between the solutions to the problem and the existence of the saddle point.

Remark 1: Let $d=\|x-m\|, m\in E$ be the distance between the point x and the convex set E Then we obtain a weak minimax relation by replacing "min" by "inf" because in such a case only the dual problem has solutions.

If E is a convex cone vertex in the origin, then the condition (ii) of Theorem 1 is equivalent to the following pair of conditions:

$(ii')\,(x_0^*, m) \le 0, \forall m \in\in E$

$(ii'')\,(x_0^*, x) \le \|x-l\|^2$

Here is the argument, from condition (ii) replacing

$x_0^*\, by - x_0^*\; we\;\; obtain\,(x_0^*, x-mn) \ge \|x-l\|^2, \forall m\in E, \forall n\in N$

Because E is a cone.

Therefore we cannot have $(x_0^*, m)\rangle 0$ for some element $m\in E, that\, is\, (ii')\, holds.$ Moreover from properties (ii) and (ii') it follows that

$\|x-l\|^2 \le (x_0^*, x-l) \le \|x_0^*\|.\|x-l\|$

$hence\; (x_0^*, x-l) = \|x-l\|^2 .Thus\; we\; have\; 0 \ge (x_0^*, l) = (x_0^*, x) = (x_0^*, x-l)(x_0^*, x) - \|x-l\|$

and from (ii) if m=0 , $(x_0^*, x) \ge \|x-l\|^2$

which implies property (ii'). The reciprocal is obvious.

When E is a linear space condition (ii') is equivalent to $(x_0^*, m) = 0, \forall m\in E,$

Because in this case E=-E.

It should be noted that $E\cap\left\{x\in \tilde{X} : \|x\| \le d\right\}$and it exists if and only if there exist separating hyperplanes which meet E. moreover, the set of all best approximations is convex and coincides with the intersection of the set will any separating hyperplanes. When this intersection non-empty the separating hyperplane is a supporting hyperplane and is given the equation

$(x^0, m-x) = \|x-l\|^2, m\in X.$

Now, let us study the existence of the best approximation. Let

d= $\inf\limits_{m\in E}\|(m-x, f)\|^2$.

We easily see that $\inf\limits_{m\in E}\|m-x\| = \inf\limits_{m\in E\cap\ddot{S}(x:d+\varepsilon}\|m-x\|$, (5)

where $\ddot{S}(x : d+\varepsilon) = \{y\in X : \|y-x\| \le d+\varepsilon\}, \varepsilon \ge 0.$

Theorem 2: *A proper convex function* $f : \tilde{X} \to]-\infty,+\infty]$ *is lower-semicontinuous on X if and only if it is lower-semicontinuous with respect to the weak topology on X.*

Proof: A fuzzy normed linear space is (strongly) closed if and only if it is closed in the corresponding weak topology on $\tilde{X}$. In particular epi f is (strongly) closed if it is weakly closed. This establish the theorem.

Theorem 3: *If the convex set E is such that* $\exists\, \varepsilon > 0$ *for which the set* $E \cap \bar{S}(x; d+\varepsilon)$ *is weakly compact, then x has a best approximation in E.*

Proof: According to relation (11) it is sufficient to recall that a lower-semi-continuous (by theorem 2) on the weakly compact set $E \cap \bar{S}(x; d+\varepsilon)$.

Corollary 2: *In a semi-reflexive fuzzy normed linear space every element possesses at least one best approximation with respect to every closed convex set.*

Proof: The set $E \cap \bar{S}(x; d+1)$ is convex closed and bounded and hence it is weakly compact by virtue of Alaoglu theorem ([5], p. 15).

Corollary 3: *In a fuzzy normed linear space every element possesses at least one best approximation with respect to every closed, convex and finite dimensional set.*

Proof: In a finite dimensional space the bounded closed convex set are compact and hence weakly compact.

Definition 20: Let $\tilde{X}, \tilde{Y}$ be fuzzy normed linear spaces of the same nature. A linear operator T: $\tilde{X} \to \tilde{Y}$ is continuous if and only if it is bounded. In other words there exits k>0 such that $\|Tl\| \le k\|l\|, \forall l \in \tilde{X}$.

The set $L(\tilde{X}, \tilde{Y})$ of all linear continuous operators defined on $\tilde{X}$ with values in $\tilde{Y}$ becomes a fuzzy normed linear space by

$$\|T\| = \sup\{\|T\,l\| : \|l\| \le 1\}$$

$$= \inf\{k : \|T\,l\| \le k\|l\|, \forall\, l \in \tilde{X}\}. \tag{6}$$

If $\tilde{Y} = R$, *we find that* $\tilde{X}^* = L(\tilde{X}, R)$, called the dual of $\tilde{X}$, is fuzzy normed linear space defined by

$$\|l^*\| = \sup\{l^*(l) : \|l\| \le 1\} \tag{7}$$

Theorem 4: *Let* f_0 *be a continuous linear functional on a linear subspace A of a fuzzy normed linear space* $\tilde{X}$ *. Then there exists a linear functional f on the whole of* $\tilde{X}$ *, i.e.,* $f \in \tilde{X}^*$, *such that*

(i) *f*/A = f_0,

(ii) $\|f\| = \|f_0\|$

Proof: Since f_0 is continuous on A, by relation (7) we have

$$f_0(m) \le \|f_0\| . \|m\|, \forall\, m \in A$$

By the Hahn-Banach Theorem for f_0 and for the convex function

$$\text{P(x)} = \{\|f_0\|.\|l\|\}$$

A specialization of this theorem yields a whole class of existence results. In this context we shall present a general and classical theorem concerning the existence of continuous linear functionals with important consequence in the duality theory of fuzzy normed linear spaces.

Theorem 5: *Let m be a nonnegative number and let h: B$\to R$ be a given real function, where B is a non-empty set of fuzzy normed linear space $\tilde{X}$. Then, h has a continuous linear extension f on all of* $\tilde{X}$ *such that* $\|f\| \le m$ *if and only if the following condition holds*:

$$\left| \sum_{i=1}^{n} \chi_i h(a_i) \right| \le m \left\lfloor \sum_{i=1}^{n} \chi_i a_i \right\rfloor, \forall n \in N, \chi_i \in R, a_i \in B.. \tag{8}$$

Proof: From relation (6) and (7) it is clear that condition (8) is necessary. To prove the sufficiency we consider A= span B and we define f_0 on A by

$$f_0(m) = \sum_{i=1}^{n} \lambda_i h(a_i),\ if\ \ m = \sum_{i=1}^{n} \lambda_i a_i \in A,\ a_i \in B\,.$$

First, using condition (8) we observed that f_0 is well defined on A. Moreover from condition (8) the continuity of f_0 on A follows and $\|f_0\| \le m$. Thus any extension given under theorem 4 has all the required properties.

Theorem 6: *For any linear subspace A of a fuzzy normed linear space*

$$\tilde{X}\ and\ l \in \tilde{X}\ there\, exists\ f \in \tilde{X}^*\ with\, the\ following\ properties:$$

$$i)\ f / A = 0,$$

$$ii)\ f(l) = \inf_{m \in A} \{\|l - m\|^2 = d(l, A)\},$$

$$iii) \|f\| = \inf_{m \in A} \{\|l - m\|\} = d(l, A).$$

Proof: We take $B = A \cup \{l\}\, and\ h : B \to R\, defined\ by\, h(m) = 0, m \in A, and\ \ h(l) = d^2(l, A).$
We observe that for any $\lambda \neq 0$ we have

$$\left|\lambda h(l) + \sum_{i=1}^{n} \lambda_i h(a_i)\right| = |\lambda h(l)| = |\lambda| d^2(l, A) \le |\lambda| d(l, A) \left\| l + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} a \right\|$$

$$= d(l, A) \left\| l + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} a \right\|, \forall n \in N, \lambda_i \in R, a_i \in B,$$

which is just inequality (8).The desire result then follows by applying Theorem 5. Indeed, we have properties (i) and (ii) and $\|f\| \le d(l,A)$. *Since* $m = d(l,A)$. On the other hand, if we consider a sequence $\{m_n\} \subset A$ such that $\|l + m_n\| \to d(l,A)$ we obtain

$$\|f\| \ge f\left(\frac{l+m_n}{\|l+m_n\|}\right) = \frac{f(l)}{\|l+m_n\|} = \frac{d^2(l,A)}{\|l+m_n\|} \to d(l,A)$$

which implies $\|f\| \ge d(l,A)$. Hence property (iii) also holds.

Corollary 4: *In a fuzzy normed linear space* $\tilde{X}$ *for every* $l \in \tilde{X}$ *there exists a continuous linear function* $f \in X^*$ *such that*

$(i)\ f(l) = \|l\|,$

$(ii)\ \|f\| = \|l\|,$

Moreover,if $l \ne 0$, there exists $g \in X^*$ such that

$(i')\ g(l) = \|l\|,$

$(ii')\ \|g\| = 1$

Proof: By Theorem 6, $d(l,A) = \|l\|$ where A={0}. Then the corollary completes the proof.

Definition 21: The space $\tilde{X}$ is strictly convex if every point of the polar set $\{l \in X : \|l\| = 1\}$ is an extreme point.

Theorem 7: *A fuzzy normed linear space* $\tilde{X}$ *is strictly convex if and only if the following equivalent properties hold*:

(i) *if* $\|x+y\| = \|x\| + \|y\|$ *and* $x \ne 0$ *there is* $t > 0$ *such that* $y = tx$

(ii) *if* $\|x\| = \|y\| = 1$ *and* $x \ne y$, *then* $\|\lambda x + (1-\lambda)y\| < 1$ *for all* $\lambda \in [0,1]$,

(iii) *if* $\|x\| = \|y\| = 1$ *and* $x \ne y$, *then* $]\frac{1}{2}(x+y)] < 1$,

(iv) *the function* $x \to \|x\|^2, x \in \tilde{X}$, *is strictly convex.*

Proof: Let $\tilde{X}$ be strictly convex and let x, y∈ $\tilde{X} \setminus \{0\}$ be such that

$$\|x+y\| = \|x\| + \|y\|$$

By corollary 4 for x∈ $\tilde{X}$, there exists a continuous functional

$x^* \in X^*$ such that $(x+y, x^*) = \|x+y\|, \leq \|x^*\| = 1$.

$(\mathrm{x}, \mathrm{x}^*) \leq \|x\|$ *and* $(y, x^*) = \|y\|$, *ie.*

$$\left(\frac{x}{\|x\|}, x^*\right) = \left(\frac{y}{\|y\|}, x^*\right) = 1$$

Because $\tilde{X}$ *is strictly* convex it follows that $\frac{x}{\|x\|} = \frac{y}{\|y\|}$

Hence property (i) holds with $t = \frac{\|y\|}{\|x\|}$

To prove that $(\mathrm{i}) \to (\mathrm{ii})$ we assume by contradiction that there exists $x \neq y$ such that $\|x\| = \|y\| = 1$ *and* $\|\lambda x + (1-\lambda) y\| = 1$.

Therefore we have $\|\lambda x + (1-\lambda) y\| = \|\lambda x\| + \|(1-\lambda) y\|$.

According to property (i) there exit $t \geq 0$ such that $\lambda x = t(1-\lambda) y$.

Since $\|x\| = \|y\|$, *we obtain* $\lambda = t(1-\lambda)$ and so x=y, which is a contradiction. The implications $(\mathrm{ii}) \to (\mathrm{iii})$ and $(\mathrm{iv}) \to (\mathrm{ii})$ are obvious.

Now we assume that $\tilde{X}$ is not strictly convex. Therefore there exist $x_0^* \in \tilde{X}^*$ and

$x_1, x_2 \in \tilde{X}$ *with* $\|x^*\| = 1, \|x_1\| = \|x_2\| = 1, x_1 \neq x_2$ *such that* $(x_1, x_0^*) = (x_2, x_0^*) = 1$, hence $(\frac{1}{2}(x_1 + x_2), x_0^*) = 1$.

Thus $\left\|\frac{1}{2}(x_1 + x_2)\right\| = \sup\{\frac{1}{2}(x_1 + x_2), x^*) : \|x^*\| = 1\} \geq (\frac{1}{2}(x_1 + x_2), x_0^*) = 1$, contradicts the property

(iii). Hence property (iii) implies the strict convexity of $\tilde{X}$. Now from the equality

$$\lambda\|x\|^2 + (1-\lambda)\|y\|^2 = \{\lambda\|x\| + (1-\lambda)\|y\|^2 = \{\lambda\|x\| + (1-\lambda)\|y\|\}^2 + \lambda(1-\lambda)(\|x\| + \|y\|)^2$$

It follows that

$\|\lambda x + (1-\lambda) y\|^2 \leq \{\lambda\|x\| + (1-\lambda)\|y\|\}^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2, \forall x, y \in \tilde{X}$

with $\|x\| \neq \|y\|$ *and* $\lambda \in]0,1[$. *If* $\|x\| = \|y\|$,

We obtain the strict convexity of the function $x \to \|x\|^2, x \in \tilde{X}$, from (ii). Thus the implication $(\mathrm{ii}) \to (\mathrm{iv})$ is established and the proof is complete.

Theorem 8: *If $\tilde{X}.$ is a fuzzy linear space which is strictly convex, then each element $x \in \tilde{X}$ possesses at most one best approximation with respect to a convex set $E \subset \tilde{X}$.*

Proof: Assume by contradiction that there exist two distinct best approximations l_1, l_2 in E. Since the set of best approximations is convex, it follows that $\frac{1}{2}(l_1 + l_2)$ is also a best approximation.

Hence if $d = \inf\{\|m - x\| : m \in E\}$, we have $0 \prec d = \|x - l_1\| = \|x - l_2\| = \left\|x - \frac{1}{2}(l_1 + l_2)\right\|$

Where $\tilde{X}.*$ is the conjugate space of $\tilde{X}.$ and $\left\|\frac{1}{d}(x - l_1)\right\| = \left\|\frac{1}{d}(x - l_2)\right\| = 1.$

In view of the strict convexity (see Theorem 7) we have

$$1 \succ \left\|\frac{1}{2d}(x - l_1) + \frac{1}{2d}(x - l_{2)}\right\| = \frac{1}{d}\left\|(x - \frac{1}{2}(l_1 + l_2)\right\| = 1,$$

which is a contradiction. This completes the proof.

Remark 2: This property is characteristic of the strictly convex spaces: if in a fuzzy normed linear space $\tilde{X}.$, every element possesses at most a best approximation with respect to every convex set (it is enough for the segments), then $\tilde{X}.$ is strictly convex.

Indeed, if we assume that $\tilde{X}.$ is not strictly convex, then there exists $x, y \in \tilde{X}, x \neq y, with\ \|x\| = \|y\| = \frac{1}{2}\|x + y\| = 1.$
Furthermore, $\|\alpha x + (1 - \alpha)y\| = 1, \forall \alpha \in [0, 1].$

Hence the origin has at the best approximation with respect to the closed convex set [x,y] every element of the set and this clearly contradicts the uniqueness.

From Corollary 2 and Theorem 3 it follows that in a semireflexive strictly fuzzy linear space, for every closed convex set E we can define the function $P_E : \tilde{X} \to \tilde{X}$ by $P_E(x) = l$, where l is the best approximation of $\tilde{X}$ by elements of E. This function is called the projection function of the space $\tilde{X}.$ into E. We note that $P_E(x) \in E$ *for every* $x \in \tilde{X}.$

Definition 22: Let us consider the following general family of minimization problems

$$(P_y)\min\{F(x, y) : x \in \tilde{X}, y \in \tilde{Y}\ where\ \tilde{X}, \tilde{Y} are\ fuzzy\ linear\ spaces\ and\ \tilde{X} \times \tilde{Y} \to \overline{R}.$$

Let us denote by

$h(y) = \inf\{F(x, y) : x \in \tilde{X}, y \in \tilde{Y}\ where\ \tilde{Y} = Y \times [0,1]$ *and* Y is a linear space over the same field K and H= $\{(y, a) \in \tilde{Y} \times R :$ *there exists* $\bar{x} \in \tilde{X}$ such that $F(\bar{x}, y) \leq a\}.$

Theorem 9: *Let* $\widetilde{X},\widetilde{Y}$ *are fuzzy linear spaces and F:* $\widetilde{X} \times \widetilde{Y} \to \left]-\infty, +\infty\right]$ *by a positively homogeneous and lower-semicontinuous function satisfying the following convexity condition*

F(x,0)>0 for any x∈ $\widetilde{X} \setminus \{0\}$. (9)

Then if epiF is locally compact, every problem P_y *has an optimal solution whenever its value is finite.*

Proof: It is easy to observe that

$$\text{H} = P_{YxR}(\text{epiF}).$$

By hypothesis epi h is a closed cone and so $(epiF)_\infty$ = epi F. Therefore, it is sufficient to use corollary 11.13 ([1], p. 28) for T= $\Pr oj_{Y\times R}$ and A = epiF, taking into account that the separation codition (1.42) of corollary 1.13 ([1], p. 28) may be written as codition (9).

Theorem 10: *If E is a closed locally compact convex set of a strictly convex fuzzy linear space* $\widetilde{X}$*, then the projection function is continuous on* $\widetilde{X}$.

Proof: If $x_n \to x, \forall \varepsilon > 0, \text{ then } \exists\, n_0(\varepsilon) \in N \text{ such that } \|x_n - x\| < \varepsilon, \forall n > n_0(\varepsilon).$ Denote

$$d_n = \inf_{m\in E}\{\|x_n - m\|\}, \quad d = \inf_{m\in E}\{\|x - m\|\},$$

where $\widetilde{X}^*$ is the dual space $\widetilde{X}$. We have

$$d_n \le \inf_{m\in E}\{\|x - m\| + \|x_n - x\|\} \prec d + \varepsilon, \ \forall n \succ n_0(\varepsilon).$$

Hence $\|x - P_E x_n\| \le \|x_n - P_E x_n\| + \|x_n - x\| \prec d_n + \varepsilon \prec d + 2\varepsilon.$

Since the set $\text{E} \cap \overline{S}(x; d+\varepsilon)$ does not contain any half-line it follows that it is compact where $\overline{S}(x; d+\varepsilon) = \{y \in \widetilde{X} : \|y - x\| \le d + \varepsilon\}, \varepsilon > 0$ .Thus $\bigcap_{\varepsilon \succ 0} \overline{S}(x; d+\varepsilon) \cap E \ne \phi$ and any subsequence of $P_E x_n$ has a cluster point which satisfies $\|x - l\| = d$ .Because $\widetilde{X}$ is strictly convex, this point is unique and so $P_E x_n \to 1 = P_E x$ as claimed.

Definition 23: A set E is called proximinal if every element of $\widetilde{X}$ has a best approximation in E. That is, the set E is proximal if the problem $\min_{m\in E}\{\|x - m\|\}$ has a solution for every x∈ $\widetilde{X}$.

Theorem 11: *A nonempty set E of a fuzzy linear space* $\widetilde{X}$ *is proximinal if and only if epi* $\|.\| + E \times \{0\}$ *is closed in X* x *R.*

Moreover if E is a convex set which contains the origin we have

$$\min_{m\in M}\{\|x - m\|\} = \max_{x^* \in S^* \cap E^0}\{(x^*, x) - P_{E^0}(x^*)\}$$ for every $x \in \widetilde{X}, where\ E^0$ is the polar set of E [2].

Proof: Taking in Therom 2.11 (Chap. 3[1]), f= I_E ,g= - $\sup_{f\in \tilde{X}^*} |(., f)|$, $A = I$, we observe that

H = {m +x, $\sup_{f\in \tilde{X}^*} |((x, f)| + r) \in XxR; m \in E, x \in X, r \succ 0\} = epi \sup_{f\in \tilde{X}^*} |(., f)| + E\, x\{0\}$.

Remark 3: It is easy to see that epi $\sup_{f\in \tilde{X}^*} |(., f)| = cone(\bar{S}(0,1) \times \{1\}$ and so if E is cone, then $epi \sup_{f\in \tilde{X}^*} |(., f)| + E \times \{0\}$ is closed in $\tilde{X}. \times R$ if and only if S(0,1)+E is closed in $\tilde{X}$

In particular, If E is a linear subspace, denoting by $\phi_E : \tilde{X} \to \tilde{X} / E$ the canonical mapping, the above condition says that $\phi_E (\bar{S}(0,1))$ *is closed in quotient space* $\tilde{X} \setminus E$.

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