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Research Paper

Generalization of Parallelepiped Equality in 2-Normed Space

Katerina Anevska[1] and Risto Malčeski[1,*]

[1] FON University, Bul. Vojvodina bb, 1000 Skopje, Macedonia

* Corresponding author, e-mail: (risto.malceski@gmail.com)

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Abstract: *In this paper we proved several corollaries concerning the parallelepiped equality in 2-normed space. Further, we generalized the stated above in terms of Gateaux derivatives.*

Keywords: 2-normed space, 2-pre-Hilbert space, Gateaux derivatives, quasi 2-pre-Hilbert space, smooth space.

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1. Introduction

Let L be a real vector space with dimension greater than 1 and $\|\cdot,\cdot\|$ be a real function on $L\times L$ satisfying:

a) $\|a,b\|\geq 0$, for every $a,b\in L$ and $\|a,b\|=0$ if and only if the set $\{a,b\}$ is linearly dependent;

b) $\|a,b\|=\|b,a\|$, for every $a,b\in L$,

c) $\|\alpha a,b\|=|\alpha|\cdot\|a,b\|$, for every $a,b\in L$ and for every $\alpha\in\mathbf{R}$,

d) $\|a+b,c\|\leq\|a,c\|+\|b,c\|$, for every $a,b,c\in L$.

Function $\|\cdot,\cdot\|$ is called 2-*norm* on L, and $(L,\|\cdot,\cdot\|)$ is called *linear* 2-*normed space* ([7]).

Let $n>1$ be a positive integer, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be a real function on $L\times L\times L$ such that:

i) $(a,a\,|\,b)\geq 0$, for every $a,b\in L$ и $(a,a\,|\,b)=0$ if and only if a and b are linearly dependent;

ii) $(a,b\,|\,c)=(b,a\,|\,c)$, for every $a,b,c\in L$,

iii) $(a,a\,|\,b)=(b,b\,|\,a)$, for every $a,b\in L$,

iv) $(\alpha a,b\,|\,c)=\alpha(a,b\,|\,c)$, for every $a,b,c\in L$ and for every $\alpha\in\mathbf{R}$, and

v) $(a+a_1,b\,|\,c)=(a,b\,|\,c)+(a_1,b\,|\,c)$, for every $a,b,a_1,c\in L$.

Function $(\cdot,\cdot\,|\,\cdot)$ is called 2-*inner product*, and $(L,(\cdot,\cdot\,|\,\cdot))$ is called 2-*pre-Hilbert space* ([3]).

Concepts of 2-norm and 2-inner product are two dimensional analogies of concepts of norm and inner product. R. Ehret proved ([6]) that, if $(L,(\cdot,\cdot\,|\,\cdot))$ be 2-pre-Hilbert space, than $\|a,b\|=(a,a\,|\,b)^{1/2}$ defines 2-norm. So, we get 2-normed vector space $(L,\|\cdot,\cdot\|)$ and for each $a,b,c\in L$ the following equalities are true:

$$(a,b\,|\,c)=\frac{\|a+b,c\|^2-\|a-b,c\|^2}{4}, \tag{1}$$

$$\|a+b,c\|^2+\|a-b,c\|^2=2(\|a,c\|^2+\|b,c\|^2), \tag{2}$$

In fact, the equality (2) is two-dimensional analogy of parallelogram equality and is called parallelepiped equality ([1]). Further, if $(L,\|\cdot,\cdot\|)$ is vector 2-normed space such that (2) is satisfied for every $a,b,c\in L$, then (1) defines 2-inner product on L, and moreover $\|a,b\|=(a,a\,|\,b)^{1/2}$, for every $a,b\in L$.

In [5] C. Diminnie and A. White by partial derivatives of 2-functionals, characterize 2-pre-Hilbert space, i.e. they prove the following: if $(L,(\cdot,\cdot\,|\,\cdot))$ be a 2-pre-Hilbert space where the norm is defined as $\|a,b\|=(a,a\,|\,b)^{1/2}$, then

$$(a,b\,|\,c)=\lim_{t\to 0}\frac{\|a+tb,c\|^2-\|a,c\|^2}{2t}.$$

In this paper, firstly will prove an equality which is direct corollary of the parallelepiped equality and an equality which is equivalent to the parallelepiped equality.

Lemma 1: *In 2-pre-Hilbert space* $(L,(\cdot,\cdot\,|\,\cdot))$ *for every* $x,y,z,u\in L$ *is true that*

$$\|z-x,u\|^2+\|z-y,u\|^2=\tfrac{1}{2}\|x-y,u\|^2+2\|z-\tfrac{1}{2}(x+y),u\|^2. \tag{3}$$

Proof: Letting $a=z-x,\ b=z-y,\ c=u$ in the equality (2) we get the following

$$\|2z-(x+y),u\|^2+\|y-x,u\|^2=2(\|z-x,u\|^2+\|z-y,u\|^2),$$

which is equivalent to equality (3). ■

Lemma 2: *In 2-pre-Hilbert space* $(L,(\cdot,\cdot\,|\,\cdot))$ *the equality (2) is equivalent to*

$$\|a+b,c\|^4-\|a-b,c\|^4=8(\|a,c\|^2+\|b,c\|^2)\cdot(a,b\,|\,c). \tag{4}$$

Proof: Let $(L,(\cdot,\cdot\,|\,\cdot))$ be a 2-pre-Hilbert space. If we multiply the equality (2) by $\|a+b,c\|^2-\|a-b,c\|^2$ and also, use the equality (1) we get the following

$$\begin{aligned}\|a+b,c\|^4-\|a-b,c\|^4 &= 2(\|a,c\|^2+\|b,c\|^2)(\|a+b,c\|^2-\|a-b,c\|^2)\\ &= 8(\|a,c\|^2+\|b,c\|^2)\cdot(a,b\,|\,c),\end{aligned}$$

It means that (2) implies (4).

Let the equality (4) be satisfied.

If $\|a+b,c\|^2 \neq \|a-b,c\|^2$, then we divide (4) by $\|a+b,c\|^2-\|a-b,c\|^2$ and get the equality (2).

If $\|a+b,c\|^2 = \|a-b,c\|^2$, i.e. $(a,b\,|\,c)=0$, then

$$\begin{aligned}\|a+b,c\|^2+\|a-b,c\|^2 = 2\|a+b,c\|^2 &= 2(a+b,a+b\,|\,c)\\ &= 2[(a,a\,|\,c)+2(a,b\,|\,c)+2(b,b\,|)]\\ &= 2(\|a,c\|^2+\|b,c\|^2),\end{aligned}$$

i.e. the equality (2) is correct. ■

2. Generalization of the Parallelepiped Equality

Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. By lemma 2.1, [2], on $L\times L\times L$ the following functionals exist:

$$N_+(x,z)(y)=\lim_{t\to 0^+}\frac{\|x+ty,z\|-\|x,z\|}{t},\quad N_-(x,z)(y)=\lim_{t\to 0^-}\frac{\|x+ty,z\|-\|x,z\|}{t}, \tag{5}$$

and are called right-hand and left-hand Gateaux derivative, respectively of a 2-norm $\|\cdot,\cdot\|$ at (x,z) in the direction y. Therefore, exists the functional

$$g(x,z)(y)=\frac{\|x,z\|}{2}(N_-(x,z)(y)+N_+(x,z)(y)). \tag{6}$$

Further, if $N_-(x,z)(y)=N_+(x,z)(y)$, then the 2-norm $\|\cdot,\cdot\|$ is said to be Gateaux differentiable at (x,z) in the direction y and is denoted by

$$N(x,z)(y)=\lim_{t\to 0}\frac{\|x+ty,z\|-\|x,z\|}{t}.$$

Theorem 1: *Let* $(L,\|\cdot,\cdot\|)$ *be a real 2-normed space. The followings are correct:*

$$g(x,z)(x)=\|x,z\|^2,\ \text{for every } x,z\in L, \tag{7}$$

$$|\, g(x,z)(y)\,| \leq \|x,z\| \cdot \|y,z\|, \text{ for every } x,y,z \in L, \tag{8}$$

$$g(x,z)(x+y) = \|x,z\|^2 + g(x,z)(y), \text{ for every } x,y,z \in L, \tag{9}$$

$$g(\alpha x,z)(\beta y) = \alpha\beta g(x,z)(y), \text{ for every } x,y,z \in L; \alpha,\beta \in \mathbf{R} \tag{10}$$

$$\|x,z\| \frac{\|x+\lambda y,z\|-\|x,z\|}{\lambda} \leq g(x,z)(y) \leq \|x,z\| \frac{\|x+ty,z\|-\|x,z\|}{t},\ \lambda<0, t>0,\ x,y,z \in L. \tag{11}$$

Proof: Let $x,z \in L$. Then,

$$\begin{aligned} g(x,z)(x) &= \tfrac{\|x,z\|}{2}(N_-(x,z)(x) + N_+(x,z)(x)) \\ &= \tfrac{\|x,z\|}{2}\Big(\lim_{t\to 0^-} \tfrac{\|x+tx,z\|-\|x,z\|}{t} + \lim_{t\to 0^+} \tfrac{\|x+tx,z\|-\|x,z\|}{t}\Big) \\ &= \tfrac{\|x,z\|}{2}\Big(\lim_{t\to 0^-} \tfrac{(1+t)\|x,z\|-\|x,z\|}{t} + \lim_{t\to 0^+} \tfrac{(1+t)\|x,z\|-\|x,z\|}{t}\Big) \\ &= \tfrac{\|x,z\|}{2}(\|x,z\| + \|x,z\|) \\ &= \|x,z\|^2, \end{aligned}$$

i.e. the equality (7) is correct.

Since the definition of the functional g and the properties of Gateaux's derivatives, follows

$$g(x,z)(y) = \tfrac{\|x,z\|}{2}(N_-(x,z)(y) + N_+(x,z)(y)) \leq \tfrac{\|x,z\|}{2} \cdot 2N_+(x,z)(y) \leq \|x,z\| \cdot \|y,z\|,$$

for every $x,y,z \in L$, i.e. the equality (8) is correct.

Equalities (5) and (6) imply

$$\begin{aligned} g(x,z)(x+y) &= \tfrac{\|x,z\|}{2}(N_-(x,z)(x+y) + N_+(x,z)(x+y)) \\ &= \tfrac{\|x,z\|}{2}\Big(\lim_{t\to 0^-} \tfrac{\|x+t(x+y),z\|-\|x,z\|}{t} + \lim_{t\to 0^+} \tfrac{\|x+t(x+y),z\|-\|x,z\|}{t}\Big) \\ &= \tfrac{\|x,z\|}{2}\Big(2\|x,z\| + \lim_{t\to 0^-} \tfrac{\|x+\frac{t}{1+t}y,z\|-\|x,z\|}{\frac{t}{1+t}} + \lim_{t\to 0^+} \tfrac{\|x+\frac{t}{1+t}y,z\|-\|x,z\|}{\frac{t}{1+t}}\Big) \\ &= \|x,z\|^2 + \tfrac{\|x,z\|}{2}(N_-(x,z)(y) + N_+(x,z)(y)) = \|x,z\|^2 + g(x,z)(y), \end{aligned}$$

for every $x,y,z \in L$, i.e. the equality (9) is correct.
For $\alpha > 0$, by the properties of 2-norm and Gateaux derivatives follows

$$\begin{aligned} g(\alpha x,z)(y) &= \tfrac{\|\alpha x,z\|}{2}(N_-(\alpha x,z)(y) + N_+(\alpha x,z)(y)) \\ &= \alpha \tfrac{\|x,z\|}{2}(N_-(x,z)(y) + N_+(x,z)(y)) = \alpha g(x,z)(y), \end{aligned}$$

For $\alpha < 0$,

$$g(\alpha x,z)(y)=\tfrac{\|\alpha x,z\|}{2}(N_-(\alpha x,z)(y)+N_+(\alpha x,z)(y))$$
$$=-\alpha\tfrac{\|x,z\|}{2}(-N_+(x,z)(y)-N_-(x,z)(y))=\alpha g(x,z)(y).$$

For $\beta>0$, by the properties of Gateaux derivatives follows

$$g(x,z)(\beta y)=\tfrac{\|x,z\|}{2}(N_-(x,z)(\beta y)+N_+(x,z)(\beta y))$$
$$=\tfrac{\|x,z\|}{2}(\beta N_-(x,z)(y)+\beta N_+(x,z)(y))=\beta g(x,z)(y),$$

For $\beta<0$,

$$g(x,z)(\beta y)=\tfrac{\|x,z\|}{2}(N_-(x,z)(\beta y)+N_+(x,z)(\beta y))$$
$$=\tfrac{\|x,z\|}{2}(\beta N_+(x,z)(y)+\beta N_-(x,z)(y))=\beta g(x,z)(y).$$

So, for every $x,y,z\in L;\alpha,\beta\in\mathbf{R}$

$$g(\alpha x,z)(\beta y)=\alpha g(x,z)(\beta y)=\alpha\beta g(x,z)(y),$$

i.e. the equality (10) is correct.

Let $t>0$ and $x,y,z\in L$. Therefore, since the corollary 2.3, [2] follows

$$N_-(x,z)(y)\le N_+(x,z)(y)$$

So,

$$g(x,z)(y)=\tfrac{\|x,z\|}{2}(N_-(x,z)(y)+N_+(x,z)(y))\le\|x,z\|\,N_+(x,z)(y), \tag{12}$$

and since, lemma 2.1, [2] the functional

$$n(x,z,y)(t)=\tfrac{\|x+ty,z\|-\|x,z\|}{t}$$

is monotony increasing at $(0,+\infty)$. Thus, we get that for $t>0$,

$$N_+(x,z)(y)\le\tfrac{\|x+ty,z\|-\|x,z\|}{t},$$

So, by inequality (12) follows

$$g(x,z)(y)\le\|x,z\|\tfrac{\|x+ty,z\|-\|x,z\|}{t},$$

i.e. the right inequality in (11) is correct. The left inequality in (11) can be proved analogously. ■

Theorem 2: $(L,\|\cdot,\cdot\|)$ *is a 2-pre-Hilbert space if and only if* $g(x,z)(y)$ *is 2-inner product of vectors* x,y *and* z*, for every* $x,y,z\in L$.

Proof: Let $(L,\|\cdot,\cdot\|)$ be 2-pre-Hilbert space, i.e. exists 2-inner product $(\cdot,\cdot|\cdot)$ such that $\|x,y\|=(x,x|y)^{1/2}$ and $x,y,z\in L$. Thus,

$$\begin{aligned} g(x,z)(y) &= \tfrac{\|x,z\|}{2}(N_-(x,z)(y)+N_+(x,z)(y)) \\ &= \tfrac{\|x,z\|}{2}\Big(\lim_{t\to 0^-}\tfrac{\|x+ty,z\|-\|x,z\|}{t}+\lim_{t\to 0^+}\tfrac{\|x+ty,z\|-\|x,z\|}{t}\Big) \\ &= \tfrac{\|x,z\|}{2}\Big(\lim_{t\to 0^-}\tfrac{2(x,y|z)+t(y,y|z)}{\|x+ty,z\|+\|x,z\|}+\lim_{t\to 0^+}\tfrac{2(x,y|z)+t(y,y|z)}{\|x+ty,z\|+\|x,z\|}\Big) \\ &= \tfrac{\|x,z\|}{2}\Big(\tfrac{(x,y|z)}{\|x,z\|}+\tfrac{(x,y|z)}{\|x,z\|}\Big) \\ &= (x,y|z), \end{aligned}$$

It means that $g(x,z)(y)$ is 2-inner product of the vectors x,y and z.

The converse of the above stated is evident by equality (7), i.e. if $g(x,z)(y)$ is 2-inner product of the vectors x,y and z, for every $x,y,z\in L$, then the equality (7) implies $(L,\|\cdot,\cdot\|)$ is 2-pre-Hilbert space. ■

Remark 1: By Theorem 2, and if $(L,\|\cdot,\cdot\|)$ is 2-pre-Hilbert space, the following equalities are true

$$g(x,z)(y)=(x,y|z)=(y,x|z)=g(y,z)(x)$$

Thus, the equality (4) (by Lemma 2, the equalities (4) and (2) are equivalent) can be written as

$$\|x+y,z\|^4-\|x-y,z\|^4=8(\|x,z\|^2\,g(x,z)(y)+\|y,z\|^2\,g(y,z)(x)), \quad (13)$$

It means the equality (13) generalize the equality (4) in 2-normed space.

3. Quasi 2-Pre-Hilbert Space

Definition 1: *2-normed space* $(L,\|\cdot,\cdot\|)$ *where the equality (13) is satisfied is said to be quasi 2-pre-Hilbert space.*

Example 1: In [9] is proved that on the set of bounded sequences of real numbers l^∞

$$\|x,y\|=\sup_{\substack{i,j\in\mathbf{N}\\ i<j}}\left|\begin{vmatrix} x_i & x_j \\ y_i & y_j\end{vmatrix}\right|,\quad x=(x_i)_{i=1}^\infty,\ y=(y_i)_{i=1}^\infty\in l^\infty$$

defines 2-norm. It means $(l^\infty,\|\cdot,\cdot\|)$ is real 2-normed space. The vectors

$$x=(1-\tfrac{1}{2},1-\tfrac{1}{2^2},\dots,1-\tfrac{1}{2^n},\dots),\ y=(0,1-\tfrac{1}{2},1-\tfrac{1}{2^2},\dots,1-\tfrac{1}{2^{n-1}},\dots)\text{ and } z=(1,0,0,\dots,0,\dots)$$

satisfy

$$\| x+y,z \|=2\,,\ \| x-y,z \|=\tfrac{1}{2^2}\,,\ \| x,z \|=\| y,z \|=1\,,\ N_+(x,z)(y)=1=N_-(x,z)(y)\,,$$
$$N_+(y,z)(x)=1=N_-(y,z)(x)\ \text{и}\ g(x,z)(y)=1=g(y,z)(x)\,.$$

Thus,

$$\| x+y,z \|^4-\| x-y,z \|^4=16-\tfrac{1}{2^8}\neq 16=8(\| x,z \|^2\, g(x,z)(y)+\| y,z \|^2\, g(y,z)(x))\,,$$

It means 2-normed space $(l^\infty,\|\cdot,\cdot\|)$ is not quasi 2-pre-Hilbert space. ■

Remark 2: Example 1, shows that 2-normed space $(l^\infty,\|\cdot,\cdot\|)$ is not quasi 2-pre-Hilbert space. Moreover, the equality (4) is satisfied in any 2-pre-Hilbert space, and in this case, the equalities (4) and (13) are equivalent. It means that each 2-pre-Hilbert space is quasi 2-pre-Hilbert space. Thus, $(l^\infty,\|\cdot,\cdot\|)$ is not quasi 2-pre-Hilbert space. But, we already said that each 2-pre-Hilbert space is quasi 2-pre-Hilbert space. Therefore, introducing the notion of quasi 2-pre-Hilbert space is reasonable.

Definition 2 ([2]): *2-normed space $(L,\|\cdot,\cdot\|)$ is said to be smooth if for $x\neq 0$ and $z\notin V(x)$ 2-norm $\|\cdot,\cdot\|$ is Gateaux differentiable at (x,z) in the direction y.*

Lemma 3 ([2]): *2-normed space $(L,\|\cdot,\cdot\|)$ is smooth if and only if one of the following conditions are satisfied: for every $x\neq 0$ and $z\notin V(x)$*

(1) $N_+(x,z)(y+y')=N_+(x,z)(y)+N_+(x,z)(y')$.

(2) $-N_+(x,z)(-y)=N_+(x,z)(y)$. ■

Lemma 4 ([2]): If $(L,\|\cdot,\cdot\|)$ *is smooth, then the following states are equivalent:*

(1) If $\| x,z \|=\| y,z \|$, then $N_+(x,z)(y)=0$ implies $N_+(y,z)(x)=0$.

(2) If $\| x,z \|=\| y,z \|$, then $\| x+ty,z \|\geq\| x,z \|$, for every $t\in\mathbf{R}$ implies $\| y+tx,z \|\geq\| y,z \|$. ■

Theorem 3: *Every quasi 2-pre-Hilbert space $(L,\|\cdot,\cdot\|)$ is smooth.*

Proof: Let $(L,\|\cdot,\cdot\|)$ be arbitrary quasi 2-pre-Hilbert space. If $t\in\mathbf{R}$ and $x,y,z\in L$, then (13) implies

$$\| (x+ty)+y,z \|^4-\| (x+ty)-y,z \|^4=8(\| x+ty,z \|^2\, g(x+ty,z)(y)+\| y,z \|^2\, g(y,z)(x+ty))\,. \quad (*)$$

Moreover, using the equalities (9) and (10), for $t\neq 0$ we get

$$g(y,z)(x+ty)=\tfrac{1}{t}g(ty,z)(x+ty)=\tfrac{1}{t}(\| ty,z \|^2+g(ty,z)(x))=t\| y,z \|^2+g(y,z)(x) \qquad (14)$$

For $t=0$ the last equality obviously holds. Letting $t\to 0$ in the equality (*), we get

$$\| x+y,z \|^4-\| x-y,z \|^4=8(\| x,z \|^2\lim_{t\to 0} g(x+ty,z)(y)+\| y,z \|^2\, g(y,z)(x))\,. \qquad (15)$$

Further, the equalities (13) and (15) imply

$$\lim_{t\to 0} g(x+ty,z)(y) = g(x,z)(y)\,. \tag{16}$$

If we substitute x and y by $x+\frac{t}{2}y$ and $\frac{t}{2}y$, respectively into the equality (13) we get

$$\|x+ty,z\|^4 - \|x,z\|^4 = 8(\tfrac{t}{2}\|x+\tfrac{t}{2}y,z\|^2\, g(x+\tfrac{t}{2}y,z)(y) + (\tfrac{t}{2})^3\|y,z\|^2\, g(y,z)(x+\tfrac{t}{2}y))$$

Thus, using the equality (14) for $t\neq 0$, we get

$$\frac{\|x+ty,z\|-\|x,z\|}{t} = \frac{4\|x+\frac{t}{2}y,z\|^2 g(x+\frac{t}{2}y,z)(y)+t^2\|y,z\|^2 g(y,z)(x+\frac{t}{2}y))}{(\|x+ty,z\|+\|x,z\|)(\|x+ty,z\|^2+\|x,z\|^2)}$$
$$= \frac{4\|x+\frac{t}{2}y,z\|^2 g(x+\frac{t}{2}y,z)(y)+t^2\|y,z\|^2 g(y,z)(x)+\frac{t^3}{2}\|y,z\|^4}{(\|x+ty,z\|+\|x,z\|)(\|x+ty,z\|^2+\|x,z\|^2)}.$$

Finally, using the equality (16) and letting $t\to 0^+$ and $t\to 0^-$ in the last one, we get

$$N_+(x,z)(y) = \frac{g(x,z)(y)}{\|x,z\|} = N_-(x,z)(y)\,,$$

It means that 2-normed space $(L,\|\cdot,\cdot\|)$ is smooth. ■

In the following theorem we'll denote necessary and sufficient condition, quasi 2-normed space be 2-pre-Hilbert space.

Theorem 4: *2-normed space L is 2-pre-Hilbert space if and only if is quasi 2-pre-Hilbert space and the following equivalence*

$$\|x+y,z\| = \|x-y,z\| \quad\Leftrightarrow\quad g(x,z)(y)=0\,, \tag{17}$$

is succeed on L.

Proof: Let L be 2-pre-Hilbert space with 2-inner product $(\cdot,\cdot|\cdot)$. Then, by Remark 1, $g(x,z)(y) = g(y,z)(x) = (x,y\,|\,z)$ holds. Thus, the equality (13) is correct, and thus the equivalence (17) are satisfied on L.

Let suppose that L is a quasi 2-pre-Hilbert space and the equivalence (17) is succeeded on L. Then, the equality (13) implies

$$\|x+ty,z\|^4 - \|x-ty,z\|^4 = 8t(\|x,z\|^2\, g(x,z)(y) + t^2\|y,z\|^2\, g(y,z)(x))\,, \tag{18}$$

for every $x,y,z\in L$ and every $t\in\mathbf{R}$. Let be $\|x+y,z\| = \|x-y,z\|$. Then, (17) implies $g(x,z)(y)=0$. The last and the fact that (18) holds for every $t\in\mathbf{R}$, imply $g(y,z)(x)=0$. By (18) follows

$$\|x+ty,z\| = \|x-ty,z\|\text{, for every } t\in\mathbf{R}\,.$$

Thus, in quasi 2-pre-Hilbert space in which the equivalence (17) is satisfied, also satisfied is the following condition: If $\| x+y,z \| = \| x-y,z \|$, then $\| x+ty,z \| = \| x-ty,z \|$, for every $t \in \mathbf{R}$. Finally, by the Theorem 7, [2], follows L is 2-pre-Hilbert space. ■

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