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Research Paper

Construction of a Class of Quasi-Symmetric 2-Designs

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Abstract: *In this article we study proper quasi-symmetric $2-$ designs i.e. block designs having two intersection numbers x and y, where $0 < x < y$. Also, we present a construction method for quasi-symmetric $2-$ designs with block intersection numbers q and $q^2 + 1$, where q is a prime number, under certain conditions on the cardinality of point set.*

Keywords: Quasi-symmetric 2-design, 3-designs, Block intersection numbers.

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1. Introduction

The theory of design of experiments came into the literature widely through the work of R. S. Fisher and F. Yates in the early $1930'$s. They are motivated by questions of design of field experiment in agriculture. During the last several decades quasi-symmetric $2-$ designs (or quasi-symmetric designs) and its classification play an important role in the study of design theory, see for example in [3] - [10]. Many applications are found in the field of graph theory as it can be represented by strongly regular graphs. Also, lot of applications of this design have been taken place in the theory of codes, basically on self-orthogonal codes. Here we discuss about a special type of balanced incomplete block design, namely quasi-symmetric designs having two intersection numbers. This article will recommend a construction method for a proper quasi-symmetric design i.e. with two distinct non-zero intersection numbers.

We start with some necessary definitions. Let X be a finite set with v elements, called points and β be a finite family of b distinct $k-$ subset of X, called blocks. Then the pair $D = (X, \beta)$ is called a balanced incomplete block design (or $2-$design) with parameter set (v,b,r,k,λ), $(v > k \geq 3)$, if each element of X is contained in exactly r blocks, each block is of size k, and any $2-$ subset of X is contained in exactly λ blocks. This r is also known as replication parameter. For $0 \leq x < k$, x is known to be the block intersection number, if there exists $B, B' \in \beta$ such that $|B \cap B'| = x$. A symmetric design is a $2-(v,k,\lambda)$ design such that $b = \lambda_0 = v$, $r = \lambda_1 = k$ and any 2 distinct blocks intersect in λ points. Now a slight generalization in the above definition will be sufficiently broad to include all symmetric designs. That generalization has been done in Fisher's inequality by $b \geq v$. Here, we are interested for those balanced incomplete block designs having exactly 2 intersection numbers with smallest one is greater than zero. A $2-$design with exactly two intersection numbers is said to be proper quasi-symmetric design, otherwise it is known to be an improper quasi-symmetric designs. We denote these intersection numbers by x and y and assume to be $0 \leq x < y < k$. Our work is related to proper quasi-symmetric design, i.e. both the intersection numbers are positive and not equal. The block graph Γ of a quasi-symmetric design D is a graph whose vertices are blocks of D with two distinct vertices are adjacent *iff* corresponding blocks intersect in y points, which basically known as strongly regular graph [3]. Much investigation on quasi-symmetric designs are simplified by these block graphs Γ or by the complement of these graphs $\overline{\Gamma}$. Let $\bar{c}$ denote the number of triangles on any edge of $\overline{\Gamma}$, then for any fixed value of $x, y \geq 2$ and $\bar{c} \geq 0$, there exist only finitely many such designs. For further reference, the reader may consult with [2] for the basic terminology of designs theory and [11] for the results on quasi-symmetric design and strongly regular graph.

Construction of quasi-symmetric design from a multiple of symmetric designs is an easy example. The article [1] describes the construction method of some combinatorial objects. In [10], it has been shown that the number of such designs is finite provided k or $\lambda(\geq 2)$ is fixed. Again the quasi-symmetric designs with $y = \lambda$ has been studied in [6]. In [4], several necessary conditions are obtained for the existence of a quasi-symmetric design with parameter set $D(v,b,r,k,\lambda;\ x,y)$ by imposing the divisibility restrictions on $y - x$.

Our main aim of this paper is to construct a family of quasi-symmetric $2-$ designs with intersection pair $(q,\ q^2+1)$, where q is any prime number. The assumption under which our work has been developed is $v \geq 2k \geq 1$ and $\lambda > 1$.

2. Preliminaries

We recall here some of the results which are useful for the development of this paper. For further reading readers can refer to [3, 9].

Lemma 1: [5] *In a $t-(v,k,\lambda)$ design, let λ_i denote the number of blocks containing any given $i-$tuple, $i = 0,1,\ldots,t$ with $\lambda_t = \lambda$; $\lambda_0 =$ band $\lambda_1 = r$. Then,*

$$\lambda_i = \frac{(v-i)}{(k-i)}\lambda_{i+1}, i = 0,1,2,\ldots,t-1.$$

Lemma 2: [2, 4, 11] *Let D be a quasi-symmetric design, with standard parameter set $(v,b,r,k,\lambda;\ x,y)$. Then the following relations hold:*

(1) $vr = bk$ *and* $\lambda(v-1) = r(k-1)$.
(2) $k(r-1)(x+y-1) - xy(b-1) = k(k-1)(\lambda-1)$.
(3) $y - x$ *divides* $k - x$ *and* $r - \lambda$.
(4) $r(-r + kr + \lambda) = bk\lambda$.

Lemma 3: [9, 10] *Let D be a quasi-symmetric design, with standard parameter set $(v,b,r,k,\lambda;\ x,y)$ and $\lambda > 1$. Then,*

(1) $x \le \frac{k^2}{v} < \lambda$.
(2) $If\ v \ge 2k\ then\ 2x < k$.

Lemma 4: [11] *Let D be a quasi-symmetric design, with standard parameter set $(v, b, r, k, \lambda;\ x, y)$ and a is the number of blocks intersecting a given block in y number of points. Then,*

(2.1) $$a = \frac{(\lambda-1)k^2 + (r-rx+x-\lambda)k}{(x-y)y}$$

(2.2) $$b = \frac{(\lambda-1)k^2 + (r-rx+x-\lambda-xy+y)k-xy}{xy}.$$

Proof: Let us consider a block B of D and make it fixed. Then if the number of blocks intersecting with B in y points is a, the remaining $b - 1 - a$ blocks intersecting with B in x points. Then by the two way counting process, the number of pair (u, B'), where B' is a block other than B in D and $u \in B \cap B'$, is obtained by the following equation,

(2.3) $ay + (b - 1 - a)x = k(r - 1)$.

Again for (u, w, B), where $u, w \in B \cap B'$, is obtained by the following equation

(2.4) $ay(y - 1) + (b - 1 - a)x(x - 1) = k(k - l)(\lambda - 1)$.

On eliminating a from the above two equations, we get (2.2) and hence (2.1).
Consequently the result follows.

3. Main Result

In this section we give the construction of a proper quasi-symmetric design from a $3 -$ design with certain parametrical restrictions.

Theorem 1: *If D be a $3 - (v, b, r, k, \lambda)$ design having intersection numbers $0, q$ and $q^2 + 1$. Then for $v \le (q^2 - q + 1)k - 1$, we can construct a quasi-symmetric design having intersection numbers q and $q^2 + 1$, where q is any prime.*

Proof: Consider D be a $3 - (v, k, \lambda)$ design with intersection numbers $0, q$ and $q^2 + 1$. We fix an arbitrary block B in D. We define an incidence structure $D(B)$ as follows: the points of $D(B)$ are the points of D not incident with B, and the blocks of $D(B)$ are the blocks B' of D for which $|B \cap B'| = 0$. Incidence structure in $D(B)$ is the same as that of D. Therefore $D(B)$ has $v - k$ points where $v - k \ge 2$. Then we shall prove that $D(B)$ will provide a quasi-symmetric design provided that it satisfies the non-triviality condition. We claim that the only block intersection numbers in $D(B)$ are q and $q^2 + 1$, otherwise it will contradict the condition of $v \le (q^2 - q + 1)k - 1$. Let a, b, c be any three elements from D with $a,\ b \notin B$ and $c \in B$. Then $|\{B' : a, b, c \in B', B' \in D\}| = \lambda$.

Let n_1 denote the number of blocks B', for which $|B \cap B'| = q$ and n_2 denote the number of blocks B'', for which $|B \cap B''| = q^2 + 1$. Then n_1 and n_2 are related by the equations

(3.1) $n_1 + n_2 = b - \lambda\binom{k}{3}$

(3.2) $qn_1 + (q^2 + 1)n_2 = k(r - 1)$

Solving (3.1) and (3.2), we get

$$n_1 = \frac{(q^2+1)\left(b - \lambda\binom{k}{3}\right) - k(r-1)}{q^2-q+1}$$

$$n_2 = \frac{q\lambda\binom{k}{3} + k(r-1) - qb}{q^2-q+1}.$$

Now we consider B' with $a, b \in B'$ with $|B \cap B'| \neq 0$ which will either be q or q^2+1. If the intersection number be q, then $|\{B' : a, b \in B', B \cap B' \neq \emptyset\}| = \frac{n_1 k}{q}$. Similarly, for the intersection number q^2+1 is $\frac{n_2 k}{q^2+1}$.

Therefore the total number of such blocks are $\frac{k((q2+1)n_1 + qn_2)}{q(q^2+1)}$.

Thus every pair of elements is in

$|\{B' : a, b \in B', B \cap B' = \emptyset\}| = r - \frac{k((q^2+1)n_1 + qn_2)}{q(q^2+1)}$ number of blocks.

Now, certainly $v - k > (q^2 - q)k - 1$ and hence $D(B)$ is a $2 - (v - k, k, r - \frac{k((q2+1)n_1 + qn_2)}{q(q^2+1)})$ design with intersection numbers q and q^2+1. Now using the **Lemma 2.(1),** we get the value of repetition parameter for the new design

$$r_1 = \frac{(rq(q^2+1) - \left((q^2+1)n_1 + qn_2\right)k)(v-k-1)}{q(q^2+1)(k-1)}$$

and the number of blocks of D(B) is

$$b_1 = \frac{(v-k)(v-k-1)(rq(q^2+1) - \left((q^2+1)n_1 + qn_2\right)k)}{q(q^2+1)k(k-1)}$$

Clearly this satisfies the condition of Fisher's Inequality. Thus the new design has its following parametrical values

$v_1 = v - k,$

$$b_1 = \frac{(v-k)(v-k-1)(rq(q^2+1) - \left((q^2+1)n_1 + qn_2\right)k)}{q(q^2+1)k(k-1)},$$

$$r_1 = \frac{(rq(q^2+1) - \left((q^2+1)n_1 + qn_2\right)k)(v-k-1)}{q(q^2+1)(k-1)},$$

k − the block length will remain same,

$\lambda_1 = \frac{k((q^2+1)n_1 + qn_2)}{q(q^2+1)}$, $x = q$ and $y = q^2 + 1$

which provide a proper quasi-symmetric design. Hence we are done.

4. Conclusion

One can construct a family of proper quasi-symmetric design having intersection numbers q and $q^2 + 1$, where q is any prime. The parametric characterization of this family of quasi-symmetric design is still remained unsolved.

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