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Research Paper

An Extension Formulas for the First Kind of Lauricella's Function of Three Variables

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Abstract: *The aim of this paper is to establish an extension formulas for the first kind of Lauricella's functions of three variables. The results are obtained with the help of two interesting special cases of the generalizations of Watson's summation theorem given by Lavoie et al.[5]. Some interesting special cases are also presented.*

Keywords: Lauricella's function, Watson's summation theorem, Exton's hypergeometric series.

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1. Introduction

Lauricella Function $F_A^{(n)}$

The Lauricella's function $F_A^{(n)}$ is defined by [6, p.60]

$$F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n)$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \qquad (1.1)$$

$$|x_1| + \cdots\cdots + |x_n| < 1$$

Pochhammer's Symbol

The Pochhammer's symbol $(a)_n$ is defined by

$$(a)_n = \begin{cases} 1 & , \quad if \quad n=0 \\ a(a+1)(a+2)....(a+n-1) & , if \quad n=1,\ 2,\ 3,... \end{cases} \tag{1.2}$$

Legendre's Duplication Formula

$$(a)_{2n} = 2^{2n}\left(\tfrac{1}{2}a\right)_n\left(\tfrac{1}{2}a+\tfrac{1}{2}\right)_n, \quad n=0,1,2,\cdots \tag{1.3}$$

Also, we note that

$$(2n)! = 2^{2n}\left(\tfrac{1}{2}\right)_n n! \tag{1.4}$$

$$(2n+1)! = 2^{2n}\left(\tfrac{3}{2}\right)_n n! \tag{1.5}$$

Exton's Double Hypergeometric Series

The Exton's double hypergeometric series $X_{E:G;H}^{A:B;D}[x,y]$ is defined by [3, p.137]

$$X\begin{matrix} A:B;B' \\ C:D;D' \end{matrix}\left[\begin{matrix} (a):(b);(b'); \\ (c):(d);(d'); \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty} \frac{((a))_{2m+n}((b))_m((b'))_n\ x^m\ y^n}{((c))_{2m+n}((d))_m((d'))_n\ m!\,n!} \tag{1.6}$$

where $((a))_m = (a_1)_m \cdots (a_A)_m$.

Watson's Summation Theorem

The classical Watson's summation theorem is given by [2, p.16]

$$ {}_3F_2\left[\begin{matrix} a,b,c & ; \\ \frac{1}{2}(a+b+1),2c & ; \end{matrix}\ 1\right]$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.7}$$

provided, $\{R(2c-a-b) > -1\}$.

Several authors investigated many generalizations and applications of the above mentioned Watson's summation theorem. For this, we refer to the research papers [1], [4] and [5]. In 1992, Lavoie et al. [5] have given the generalizations of the classical Watson's theorem on the sum of a ${}_3F_2(1)$ in the following form:

$$ {}_3F_2\left[\begin{matrix} a,b,c & ; \\ \frac{1}{2}(a+b+i+1),2c+j & ; \end{matrix}\ 1\right]$$

$$= A_{i,j}\,\frac{2^{a+b+i-2}\Gamma(\tfrac{1}{2}(a+b+i+1))\,\Gamma(c+[\tfrac{j}{2}]+\tfrac{1}{2})\,\Gamma(c-\tfrac{1}{2}(a+b-j-1)-\tfrac{1}{2}|i+j|)}{\Gamma(\tfrac{1}{2})\Gamma(a)\Gamma(b)}$$

$$\times\left\{B_{i,j}\,\frac{\Gamma(\tfrac{1}{2}a+\tfrac{1}{4}(1-(-1)^{i}))\Gamma(\tfrac{1}{2}b)}{\Gamma(c-\tfrac{1}{2}a+\tfrac{1}{2}+[\tfrac{j}{2}]-\tfrac{(-1)^{j}}{4}(1-(-1)^{i}))\Gamma(c-\tfrac{1}{2}b+\tfrac{1}{2}+[\tfrac{j}{2}])}\right.$$

$$\left.+C_{i,j}\,\frac{\Gamma(\tfrac{1}{2}a+\tfrac{1}{4}(1+(-1)^{i}))\Gamma(\tfrac{1}{2}b+\tfrac{1}{2})}{\Gamma(c-\tfrac{1}{2}a+[\tfrac{j+1}{2}]+\tfrac{(-1)^{j}}{4}(1-(-1)^{i}))\Gamma(c-\tfrac{1}{2}b+[\tfrac{j+1}{2}])}\right\} \quad (1.8)$$

$$\text{for } (i, j = 0, \pm 1, \pm 2).$$

Also $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $A_{i,j}$ $B_{i,j}$ and $C_{i,j}$ are given respectively in [5]. For $i = j = 0$, (1.8) reduces immediately to the classical Watson's theorem (1.7).

In 2007, Kim and Rathie [4], have given the followingtwospecial cases of (1.8):

$${}_3F_2\left[\begin{matrix}-2n, 1-2c-i-2n, d & ; \\ 1-c-2n, 2d+j & ;\end{matrix}\ 1\right]$$

$$= D_{i,j}\,\frac{(\tfrac{1}{2})_n(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}-\tfrac{1}{8}+\tfrac{(-1)^{i}}{8}+\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1-(-1)^{i})])_n}{(d+\tfrac{1}{2}+[\tfrac{j}{2}])_n(c+d+\tfrac{i}{2}-\tfrac{1}{4}+\tfrac{(-1)^{i}}{4}+[\tfrac{j}{2}+\tfrac{1}{4}(1-(-1)^{i})])_n}$$

$$\times\frac{(c+\tfrac{1}{2}+\tfrac{i}{2}-\tfrac{1}{4}(1+(-1)^{i}))_n(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}+\tfrac{3}{8}+\tfrac{(-1)^{i}}{8}+\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1-(-1)^{i})])_n}{(\tfrac{1}{2}c+\tfrac{1}{4}+\tfrac{i}{4}-\tfrac{1}{8}(1+(-1)^{i}))_n(\tfrac{1}{2}c+\tfrac{3}{4}+\tfrac{i}{4}-\tfrac{1}{8}(1+(-1)^{i}))_n} \quad (1.9)$$

$${}_3F_2\left[\begin{matrix}-2n-1, -2c-i-2n, d & ; \\ -c-2n, 2d+j & ;\end{matrix}\ 1\right]$$

$$= E_{i,j}\,\frac{(\tfrac{3}{2})_n(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}+\tfrac{1}{8}-\tfrac{(-1)^{i}}{8}-\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1+(-1)^{i})])_n}{(d+\tfrac{1}{2}+[j+\tfrac{1}{2}])_n(c+d+\tfrac{i}{2}+\tfrac{1}{4}-\tfrac{(-1)^{i}}{4}+[\tfrac{j}{2}+\tfrac{1}{4}(1+(-1)^{i})])_n}$$

$$\times\frac{(c+\tfrac{3}{2}+\tfrac{i}{2}-\tfrac{1}{4}(3-(-1)^{i}))_n(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}+\tfrac{5}{8}-\tfrac{(-1)^{i}}{8}-\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1+(-1)^{i})])_n}{(\tfrac{1}{2}c+\tfrac{3}{4}+\tfrac{i}{4}-\tfrac{1}{8}(3-(-1)^{i}))_n(\tfrac{1}{2}c+\tfrac{5}{4}+\tfrac{i}{4}-\tfrac{1}{8}(3-(-1)^{i}))_n} \quad (1.10)$$

for $i, j = 0, \pm 1, \pm 2$

The coefficients $D_{i,j}$ *and* $E_{i,j}$ can be obtained from the tables of $D_{i,j}$ *and* $E_{i,j}$ given in [5] by replacing a by $1-2c-i-4n$ and $-1-2c-i-4n$ respectively.

The main object of this researchpaper isto establish an extension formula for the first kind of Lauricella's functions of three variables. The results are obtained with the help of the results (1.9) and (1.10).

2. Main Result

The following formula for $F_A^{(3)}$ will be established in this section:

$$F_A^{(3)}\left(a,b,c,d\ ;e,\ 2c+i,\ 2d+j\ ;\ x\ ,y,-y\right)$$

$$=\sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}D_{i,j}\ \frac{(a)_{m+2n}\ (b)_m\ (\tfrac{1}{2}c)_n(\tfrac{1}{2}c+\tfrac{1}{2})_n x^m (y^2/4)^n}{(e)_m(c+\tfrac{i}{2})_n(c+\tfrac{1}{2}+\tfrac{i}{2})_n\ m!n!}$$

$$\times\frac{(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}-\tfrac{1}{8}+\tfrac{(-1)^i}{8}+\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1-(-1)^i)])_n}{(d+\tfrac{1}{2}+[\tfrac{j}{2}])_n(c+d+\tfrac{i}{2}-\tfrac{1}{4}+\tfrac{(-1)^i}{4}+[\tfrac{j}{2}+\tfrac{1}{4}(1-(-1)^i)])_n}$$

$$\times\frac{(c+\tfrac{1}{2}+\tfrac{i}{2}-\tfrac{1}{4}(1+(-1)^i))_n(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}+\tfrac{3}{8}+\tfrac{(-1)^i}{8}+\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1-(-1)^i)])_n}{(\tfrac{1}{2}c+\tfrac{1}{4}+\tfrac{i}{4}-\tfrac{1}{8}(1+(-1)^i))_n(\tfrac{1}{2}c+\tfrac{3}{4}+\tfrac{i}{4}-\tfrac{1}{8}(1+(-1)^i))_n}$$

$$+\sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}E_{i,j}\left(\frac{acy}{2c+i}\right)\frac{(a+1)_{m+2n}\ (b)_m\ (\tfrac{1}{2}c+\tfrac{1}{2})_n(\tfrac{1}{2}c+1)_n x^m (y^2/4)^n}{(e)_m(c+\tfrac{i}{2}+\tfrac{1}{2})_n\ (c+\tfrac{i}{2}+1)_n\ m!n!}$$

$$\times\frac{(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}+\tfrac{1}{8}-\tfrac{(-1)^i}{8}-\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1+(-1)^i)])_n}{(d+\tfrac{1}{2}+[j+\tfrac{1}{2}])_n(c+d+\tfrac{i}{2}+\tfrac{1}{4}-\tfrac{(-1)^i}{4}+[\tfrac{j}{2}+\tfrac{1}{4}(1+(-1)^i)])_n}$$

$$\times\frac{(c+\tfrac{3}{2}+\tfrac{i}{2}-\tfrac{1}{4}(3-(-1)^i))_n(\tfrac{1}{2}c+\tfrac{1}{2}d+\tfrac{i}{4}+\tfrac{5}{8}-\tfrac{(-1)^i}{8}-\tfrac{1}{2}[\tfrac{j}{2}+\tfrac{1}{4}(1+(-1)^i)])_n}{(\tfrac{1}{2}c+\tfrac{3}{4}+\tfrac{i}{4}-\tfrac{1}{8}(3-(-1)^i))_n(\tfrac{1}{2}c+\tfrac{5}{4}+\tfrac{i}{4}-\tfrac{1}{8}(3-(-1)^i))_n} \quad (2.1)$$

for $i,\ j=0,\ \pm 1,\ \pm 2$.

Proof of (2.1): Denoting the left hand side of (2.1) by S, expanding $F_A^{(3)}$ in a power series and using the results

$$(a)_{m+n}=\ (a)_m\,(a+m)_n \quad (2.2)$$

$$\sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}A(n,m)=\sum_{m=0}^{\infty}\ \sum_{n=0}^{m}A(n,m-n)\,, \quad (2.3)$$

we get

$$S=\sum_{m=0}^{\infty}\frac{(a)_m\ (b)_m\ x^m}{(e)_m m!}\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(a+m)_n\ \ (c)_{n-p}(d)_p(-1)^p\, y^n}{(2c+i)_{n-p}(2d+j)_p\,(n-p)!\,p!} \quad (2.4)$$

Next, using the identities

$$(a)_{m-n}=\frac{(-1)^n(a)_m}{(1-a-m)_n},0\le n\le m \quad \text{and} \quad (m-n)!=\frac{(-1)^n\,m!}{(-m)_n},\ 0\le n\le m$$

(2.5)

we get

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m\,(c)_n\;x^m\,y^n}{(e)_m\,(2c+i)_n\;m!n!}\,{}_3F_2\left[\begin{matrix}-n\,,1-2c-i-n,d\;;\\ 1-c-n,2d+j\;;\end{matrix}\,1\right] \tag{2.6}$$

Separating (2.6) into its even and odd termsby using the elementary identity

$$\sum_{n=0}^{\infty}A(n)=\sum_{n=0}^{\infty}A(2n)+\sum_{n=0}^{\infty}A(2n+1)\,, \tag{2.7}$$

we get

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n}\,(b)_m\,(c)_{2n}\;x^m\,y^{2n}}{(e)_m\,(2c+i)_{2n}\;m!(2n)!}\,{}_3F_2\left[\begin{matrix}-2n\,,1-2c-i-2n,d\;;\\ 1-c-2n,2d+j\;;\end{matrix}\,1\right]$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n+1}\,(b)_m\,(c)_{2n+1}\;x^m\,y^{2n+1}}{(e)_m\,(2c+i)_{2n+1}\;m!(2n+1)!}\,{}_3F_2\left[\begin{matrix}-2n-1\,,-2c-i-2n,d\;;\\ -c-2n,2d+j\;;\end{matrix}\,1\right] \tag{2.8}$$

Now, using the results (1.3), (1.4) and (1.5), we get

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n}\,(b)_m\,(\tfrac{1}{2}c)_n(\tfrac{1}{2}c+\tfrac{1}{2})_n\,x^m(y^2/4)^n}{(e)_m(c+\tfrac{i}{2})_n(c+\tfrac{1}{2}+\tfrac{i}{2})_n\,(\tfrac{1}{2})_n\,m!n!}\,{}_3F_2\left[\begin{matrix}-2n,\,1-2c-i-2n,d\;;\\ 1-c-2n,\,2d+j\;;\end{matrix}\,1\right]$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{acy}{2c+i}\right)\frac{(a+1)_{m+2n}\,(b)_m\,(\tfrac{1}{2}c+\tfrac{1}{2})_n(\tfrac{1}{2}c+1)_n\,x^m(y^2/4)^n}{(e)_m(c+\tfrac{i}{2}+\tfrac{1}{2})_n\,(c+\tfrac{i}{2}+1)_n\,(\tfrac{3}{2})_n\,m!n!}$$

$$\times{}_3F_2\left[\begin{matrix}-2n-1\,,-2c-i-2n,d\;;\\ -c-2n,2d+j\;;\end{matrix}\,1\right] \tag{2.9}$$

Finally, in (2.9) if weuse (1.9) and (1.10), then we arrive at the right hand side of (2.1). This completes the proof of (2.1).

3. Special Cases

Taking $i=j=0$ in (2.1), we get

$$F_A^{(3)}\left(a,b,c,d\;;e,\,2c,\,2d\;;\,x\,,y,-y\right)$$
$$=X\,{}^{1:2;1}_{0:3;1}\left[\begin{matrix}a\;:\tfrac{1}{2}(c+d),\tfrac{1}{2}(c+d+1);b;\\ -\;:\;c+\tfrac{1}{2},d+\tfrac{1}{2},c+d\;\;;e;\end{matrix}\;\frac{y^2}{4},x\right] \tag{3.1}$$

Further, taking $c=d$ in (3.1), we get

$$F_A^{(3)}\left(a,b,c,c\;;e,\,2c,\,2c\;;\,x\,,y,-y\right)$$

$$= X^{1:1;1}_{0:2;1}\left[\begin{matrix} a & : & c & ;b\,; \\ - & : & 2c, c+\frac{1}{2} & ;e\,; \end{matrix} \;\frac{y^2}{4}, x\right] \tag{3.2}$$

Taking $i=1,\ j=0$ in (2.1), we get

$$F_A^{(3)}\left(a,b,c,d\,;e,\,2c+1,\,2d\,;\,x\,,y,-y\right)$$

$$= X^{1:2;1}_{0:3;1}\left[\begin{matrix} a & :\frac{1}{2}(c+d), \frac{1}{2}(c+d+1); b; \\ - & :\ c+\frac{1}{2}, d+\frac{1}{2}, c+d \quad ;e; \end{matrix} \;\frac{y^2}{4}, x\right]$$

$$-\frac{ay}{2(2c+1)} X^{1:2;1}_{0:3;1}\left[\begin{matrix} a+1 & :\frac{1}{2}(c+d+1), \frac{1}{2}(c+d+2);\, b; \\ - & :\ c+\frac{3}{2}, d+\frac{1}{2}, c+d+1 \quad ;\, e; \end{matrix} \;\frac{y^2}{4}, x\right] \tag{3.3}$$

Further, taking $c=d$ in (3.3), we get

$$F_A^{(3)}\left(a,b,c,c\,;e,\,2c+1,\,2c\,;\,x\,,y,-y\right)$$

$$= X^{1:1;1}_{0:2;1}\left[\begin{matrix} a & : & c & ;b; \\ - & : & c+\frac{1}{2}, 2c & ;e; \end{matrix} \;\frac{y^2}{4}, x\right]$$

$$-\frac{ay}{2(2c+1)} X^{1:1;1}_{0:2;1}\left[\begin{matrix} a+1 & : & c+1 & ;\, b; \\ - & : & c+\frac{3}{2},\, 2c+1 & ;\, e; \end{matrix} \;\frac{y^2}{4}, x\right] \tag{3.4}$$

Taking $i=0,\ j=1$ in (2.1), we get

$$F_A^{(3)}\left(a,b,c,d\,;e,\,2c,\,2d+1\,;\,x\,,y,-y\right)$$

$$= X^{1:2;1}_{0:3;1}\left[\begin{matrix} a & :\frac{1}{2}(c+d), \frac{1}{2}(c+d+1); b; \\ - & :\ c+\frac{1}{2}, d+\frac{1}{2}, c+d \quad ;e; \end{matrix} \;\frac{y^2}{4}, x\right]$$

$$+\frac{ay}{2(2d+1)} X^{1:2;1}_{0:3;1}\left[\begin{matrix} a+1 & :\frac{1}{2}(c+d-1), \frac{1}{2}(c+d)\,;\, b; \\ - & :c+\frac{1}{2}, d+\frac{3}{2}, c+d+1\,;\, e; \end{matrix} \;\frac{y^2}{4}, x\right] \tag{3.5}$$

The other special cases of (2.1) can be obtained by the similar manner.

4. Conclusion

We conclude our present investigation by remarking that the results established here can be applied to obtain a large number of transformation formulas for the first kind of Lauricella's function of three variables. Further, in the formula (2.1) if we take $c=d$,then we can obtain new extension formulas of Lauricella's function of three variables $F_A^{(3)}\left(a,b,c,c\,;e,\,2c+i,\,2c+j;\,x\,,y,-y\right)$. Also twenty five results as special cases of this extension formulas can also be obtained in terms of Exton's double hypergeometric series for $i, j = 0,\pm1,\pm2$.

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