Abstract: The aim of this paper is to introduce and study the notion of \( S_nP_m \)-open sets and \( S_nP_m \)-continuity such that \( \mathcal{N}, \mathcal{M} \in \{ \tau, \delta, \theta \} \). Also, some characterizations and several properties of these notions are presented. Finally we show that certain results of several publications on the concepts of weakness and strength of nearly open sets are considered as corollaries of the results of this research.

Keywords: \( S_nP_m \)-open, \( S_nP_m \)-continuous functions and \( (S_nP_m, \tau) \)-graph.

1. Introduction

The notion of Z-open (Y-open) and Z-continuity (resp. M-open and M-continuity, b-open, sp-open, \( \gamma \)-open and \( \gamma \)-continuity, e-open and e-continuity, Z*-open and Z*-continuity, z-open and z-continuity) was introduced and study by A. I. EL-Maghrabi and A.M. Mubarki [12, 11] (resp. A.I. EL-Maghrabi and M.A. AL-Juhani [8, 10], D. Andrijevi’c [1], J. Dontchev and M. Przemski [4], A.A.EL-Atik [6], E. Ekici [5], A. M. Mubarki [15], M. Ozkoc [17]) in topological spaces. The purpose of this paper is to introduce and study the notion of generalizations of nearly open sets of a type \( S_nP_m \)-open such that \( \mathcal{N}, \mathcal{M} \in \{ \tau, \delta, \theta \} \). By means of \( S_nP_m \)-open we define \( S_nP_m \)-continuity. Also, we show that some results in several papers considered as corollaries from the results of this paper.
2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) (simply. X and Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), the closure of A, the interior of A and the complement of A denoted by cl(A), int(A) and X \ A respectively. A space (X, τ) is called extremely disconnected (briefly. E. D) [20] if the closure of every open set of X is open. A subset A of a space (X, τ) is called regular open (resp. regular closed) [21] if A = int(cl(A)) (resp. A = cl(int(A))). A point x ∈ X is called a δ-adherent (resp. δ-adherent) point of A [19] if A ∩ cl(V) ≠ ∅ (resp. A ∩ int(cl(V)) ≠ ∅), for every open set V containing x. The set of all δ-adherent (resp. δ-adherent) points of A is called the δ-closure (resp. δ-closure) of A and is denoted by clδ(A) (resp. clδ(A)). A subset A of X is called δ-closed (resp. δ-closed) if A = clδ(A) (resp. A = clδ(A)). The complement of a θ-closed (resp. δ-closed) set is called θ-open (resp. δ-open). The δ-interior (resp. δ-interior) of set A in X and will denoted by intδ(A) (resp. intδ(A)) consists of those points x of A such that for some open set U containing x, cl(U) ⊆ A (resp. U ⊆ int(cl(U)) ⊆ A). A subset A of a space X is called θ-semiopen [2] (resp. δ-semiopen [18], semiopen [13], preopen [14], δ-preopen [19], θ-preopen [16], Y-open [11], Z-open [12], Z′-open [15], M-open [8], b-open [1] or γ-open [6] or sp-open [4], e-open [5], ζ-open [17]) if A ⊆ cl(intδ(A)) (resp. A ⊆ cl(intδ(A)), A ⊆ cl(int(A)), A ⊆ int(clδ(A)), A ⊆ int(clδ(A)), A ⊆ cl(intδ(A)) U int(cl(A)), A ⊆ cl(intδ(A)) U int(clδ(A)), A ⊆ cl(intδ(A)) U int(clδ(A)), A ⊆ cl(intδ(A)) U int(clδ(A)), A ⊆ cl(intδ(A)) U int(clδ(A))). The complement of a θ-semiopen (resp. δ-semiopen, semiopen, preopen, δ-preopen, Y-open, Z-open, Z′-open, M-open, b-open or γ-open or sp-open, e-open, ζ-open) sets is called θ-semi-closed [2] (resp. δ-semi-closed [18], semi-closed [3], pre-closed [14], δ-pre-closed [19], θ-pre-closed [16], Y-closed [11], Z-closed [12], Z′-closed [15], M-closed [8], b-closed [1] or γ-closed [6] or sp-closed [4], e-closed [5], ζ-closed [17]). The intersection of all θ-semi-closed (resp. δ-semi-closed, semi-closed, pre-closed, δ-pre-closed, θ-pre-closed) sets containing A is called the θ-semi-closure [2] (resp. δ-semi-closure [18], semi-closure [3], pre-closure [14], δ-pre-closure [19], θ-pre-closure [16]) of A and is denoted by sclδ(A) (resp. sclδ(A), scl(A), pcl(A), pclδ(A), pclδ(A)). The union of all θ-semiopen (resp. δ-semiopen, semiopen, preopen, δ-preopen, θ-preopen) sets contained in A is called the θ-semi-interior [2] (resp. δ-semi-interior [18], semi-interior [13], pre-interior [14], δ-pre-interior [19], θ-pre-interior [16]) of A and is denoted by sintδ(A) (resp. sintδ(A), sint(A), pint(A), pintδ(A), pintδ(A)). The collections of (open, δ-open, θ-open), (semiopen, δ-semiopen, semiopen) and (preopen, δ-preopen, θ-preopen) will be as a generalization and denoted by (n-open),(n-semiopen), (n-preopen) respectively. The family of all n-open (resp. n-semi open, n-preopen) is denoted by nO(X) (resp. nSO(X), nPO(X)).

Lemma 2.1: [1,11,13,14, 16,18, 19]. The following hold for subsets A, B of a space (X, τ).

1. pclδ(A) = A U cl(intδ(A)) and pintδ(A) = A ∩ int(clδ(A)),

2. sintδ(A) = A ∩ cl(intδ(A)) and sclδ(A) = A U int(clδ(A)),

3. sclδ(X \ A) = X \ sintδ(A), sclδ(A U B) ⊆ sclδ(A) U sclδ(B),
(4) $pcl_n(X \setminus A) = X \setminus pint_n(A)$, $pcl_n(A \cup B) \subseteq pcl_n(A) \cup pcl_n(B)$.

3. $S_n P_m$-Open Sets and $S_n P_m$-Closed Sets

**Definition 3.1:** A subset $A$ of a topological space $(X, \tau)$ is said to be:

1. $S_n P_m$-open if $A \subseteq cl(int_n(A)) \cup int(cl_m(A))$.
2. $S_n P_m$-closed if $int(cl_n(A)) \cap cl(int_m(A)) \subseteq A$.

The family of all $S_n P_m$-open (resp. $S_n P_m$-closed) subsets of a space $(X, \tau)$ will be as always denoted by $S_n P_m O(X)$ (resp. $S_n P_m C(X)$).

**Theorem 3.1:** For a space $(X, \tau)$. If every $A \subseteq X$ is nowhere dense, then $n SO(X) = S_n P_m O(X)$.

**Theorem 3.2:** Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then any $A \in S_n P_m O(X)$ is $m$-preopen if one of the following conditions is hold:

1. $(X, \tau)$ is E.D.,
2. $X \setminus A$ is dense in $X$.

**Proof:** (1) Since, $A \in S_n P_m O(X)$ and $X$ is E.D., then $A \subseteq cl(int_n(A)) \cup int(cl_m(A)) \subseteq int(cl(int_n(A))) U int(cl_m(A)) = int(cl_m(A))$. Therefore $A$ is $m$-preopen in $X$.

(2) Since, $A \in S_n P_m O(X, \tau)$ and $X \setminus A$ is dense in $X$, then $int_n(A) = \emptyset$ and $A \subseteq cl(int_n(A)) \cup int(cl_m(A)) = int(cl_m(A))$. Hence $A$ is $m$-preopen in $X$.

**Theorem 3.3:** Let $(X, \tau)$ be a topological space. Then the following are hold.

1. The arbitrary union of $S_n P_m$-open sets is $S_n P_m$-open,
2. The arbitrary intersection of $S_n P_m$-closed sets is $S_n P_m$-closed.

**Proof:** Let $\{A_i, i \in I\}$ be a family of $S_n P_m$-open sets. Then

$$\bigcup_i A_i \subseteq \bigcup_i (cl(int_n(A_i)) \cup int(cl_m(A_i))) \subseteq cl(int_n(\bigcup_i A_i)) \cup int(cl_m(\bigcup_i A_i)),$$

for all $i \in I$.

**Remark 3.1:** By the following example we show that the intersection of any two $S_n P_m$-open sets is not $S_n P_m$-open.

**Example 3.1:** (a) Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then $A = \{a, c\}$ and $B = \{b, c\}$ are $S_n P_m$-open sets. But, $A \cap B = \{c\}$ is not $S_n P_m$-open if $n \in \{\tau, \delta, \theta\}$ and $m \in \{\tau, \delta\}$.
Consider the real number space $\mathbb{R}$ with the usual topology. Then the sets $A = (-\infty, 0) \cup [1, 5]$ and $B = (-\infty, 0] \cup (1, 5]$ is a $S_{n}P_{m}$-open sets. But $A \cap B = (-\infty, 0) \cup (1, 5]$ it is not $S_{n}P_{m}$-open sets if $n \in \{\delta, \theta\}$ and $m \in \{\theta\}$.

**Problem (1):** Give an example showing that the intersection of any two $S_{\tau}P_{\theta}$-open sets is not $S_{\tau}P_{\theta}$-open

**Definition 3.2:** Let $(X, \tau)$ be a topological space. Then:

1. The union of all $S_{n}P_{m}$-open sets of $X$ contained in $A$ is called the $S_{n}P_{m}$-interior of $A$ and is denoted by $S_{n}P_{m}$-int$(A)$,

2. The intersection of all $S_{n}P_{m}$-closed sets of $X$ containing $A$ is called the $S_{n}P_{m}$-closure of $A$ and is denoted by $S_{n}P_{m}$-cl$(A)$.

**Theorem 3.4:** Let $A, B$ be two subsets of a space $(X, \tau)$. Then the following statements are hold:

1. $S_{n}P_{m}$-cl$(\emptyset) = \emptyset$ and $S_{n}P_{m}$-cl$(X) = X$,

2. $A \subseteq S_{n}P_{m}$-cl$(A)$,

3. If $A \subseteq B$, then $S_{n}P_{m}$-cl$(A) \subseteq S_{n}P_{m}$-cl$(B)$,

4. $x \in S_{n}P_{m}$-cl$(A)$ if and only if for each $S_{n}P_{m}$-open set $U$ containing $x$ such that $U \cap A \neq \emptyset$,

5. $S_{n}P_{m}$-closed if and only if $A = S_{n}P_{m}$-cl$(A)$,

6. $S_{n}P_{m}$-cl$(S_{n}P_{m}$-cl$(A)) = S_{n}P_{m}$-cl$(A)$,

7. $S_{n}P_{m}$-cl$(A) \cup S_{n}P_{m}$-cl$(A) \subseteq S_{n}P_{m}$-cl$(A \cup B)$,

8. $S_{n}P_{m}$-cl$(A) \cap S_{n}P_{m}$-cl$(A) \supseteq S_{n}P_{m}$-cl$(A \cap B)$.

**Proof:** (6) By using (2) and $A \subseteq S_{n}P_{m}$-cl$(A)$, we have $S_{n}P_{m}$-cl$(A) \subseteq S_{n}P_{m}$-cl$(S_{n}P_{m}$-cl$(A))$. Let $x \in S_{n}P_{m}$-cl$(S_{n}P_{m}$-cl$(A))$. Then, for every $S_{n}P_{m}$-open set $V$ containing $x$, $V \cap S_{n}P_{m}$-cl$(A) \neq \emptyset$. Let $y \in V \cap S_{n}P_{m}$-cl$(A)$. Then, for every a $S_{n}P_{m}$-open set $G$ containing $y$, $A \cap G \neq \emptyset$.

Since $V$ is a $S_{n}P_{m}$-open set, $y \in V$ and $A \cap V \neq \emptyset$, then $x \in S_{n}P_{m}$-cl$(A)$. Therefore, $S_{n}P_{m}$-cl$(S_{n}P_{m}$-cl$(A)) \subseteq S_{n}P_{m}$-cl$(A)$.

**Theorem 3.5:** Let $A, B$ be two subsets of a space $(X, \tau)$. Then the following statements are hold:

1. $S_{n}P_{m}$-int$(\emptyset) = \emptyset$, $S_{n}P_{m}$-int$(X) = X$,

2. $S_{n}P_{m}$-int$(A) \subseteq A$,
(3) If \( A \subseteq B \), then \( S_P m^{-}\text{int}(A) \subseteq S_P m^{-}\text{int}(B) \).

(4) \( x \in S_P m^{-}\text{int}(A) \) if and only if there exists a \( S_P \)-open set \( W \) such that \( x \in W \subseteq A \).

(5) \( A \) is \( S_P \)-open if and only if \( A = S_P m^{-}\text{int}(A) \).

(6) \( S_P m^{-}\text{int}(S_P m^{-}\text{int}(A)) = S_P m^{-}\text{int}(A) \).

(7) \( S_P m^{-}\text{int}(A) \cup S_P m^{-}\text{int}(B) \subseteq S_P m^{-}\text{int}(A \cup B) \).

(8) \( S_P m^{-}\text{int}(A \cap B) \subseteq S_P m^{-}\text{int}(A) \cap S_P m^{-}\text{int}(B) \).

**Theorem 3.6:** Let \( A \) be two subset of a topological space \((X, \tau)\). Then the following are hold.

(1) \( S_P m^{-}\text{cl}(X \setminus A) = X \setminus S_P m^{-}\text{int}(A) \).

(2) \( S_P m^{-}\text{int}(X \setminus A) = X \setminus S_P m^{-}\text{cl}(A) \).

**Proof:** It follows from the fact the complement of \( S_P m^{-}\text{open} \) set is \( S_P \)-closed and \( \bigcap_i (X \setminus A_i) = X \setminus \bigcup_i A_i \).

**Lemma 3.1:** Let \( A \) be a subset of a topological space \((X, \tau)\). Then the following are hold:

(1) \( \text{pint}_n(pcl_m(A)) = pcl_m(A) \cap \text{int}(cl_n(A)) \).

(2) \( pcl_m(\text{pint}_n(A)) = \text{pint}_m(A) \cup \text{cl}(\text{int}_n(A)) \).

**Proof:** (1) By Lemma 2.1 \( \text{pint}_n(pcl_m(A)) = pcl_m(A) \cap \text{int}(cl_n(pcl_m(A))) = pcl_m(A) \cap \text{int}(cl_n(A \cup cl(int_m(A)))) = pcl_m(A) \cap \text{int}(cl_n(A)) \).

(2) Obvious.

**Theorem 3.7:** Let \( A \) be a subset of a topological space \((X, \tau)\). Then the following are equivalent:

(1) \( A \) is a \( S_P \)-open sets,

(2) \( A = \text{sint}_n(A) \cup \text{pint}_m(A) \).

(3) \( A = A \cap pcl_n(\text{pint}_m(A)) \).

(4) \( A \subseteq pcl_n(\text{pint}_m(A)) \).

(5) there exists \( U \in \mathcal{PO}(X) \) such that \( U \subseteq A \subseteq pcl_n(U) \).
(6) $\text{pcl}_n(A) = \text{pcl}_n(\text{pint}_m(A))$.

**Proof:** (1) $\rightarrow$ (2). Obvious.
(2) $\rightarrow$ (3). It follows from Lemmas 2.1, 3.1,
(3) $\rightarrow$ (4). Obvious,
(4) $\rightarrow$ (5). It follows from putting $U = m\text{pint}(A)$,
(5) $\rightarrow$ (6). Let there exists $U \in m\text{PO}(X)$ such that $U \subseteq A \subseteq \text{pcl}_n(U)$. Since $U \subseteq A$, then

$\text{pcl}_n(U) \subseteq \text{pcl}_n(\text{pint}_m(A))$ and hence $A \subseteq \text{pcl}_n(U) \subseteq \text{pcl}_n(\text{pint}_m(A))$, so $\text{pcl}_n(A) \subseteq \text{pcl}_n(U) \subseteq \text{pcl}_n(\text{pint}_m(A))$. But $\text{pcl}_n(\text{pint}_m(A)) \subseteq \text{pcl}_n(A)$. Therefore, $\text{pcl}_n(A) = \text{pcl}_n(\text{pint}_m(A))$,

(6) $\rightarrow$ (1). By Lemma 3.1, we have $A \subseteq \text{pcl}_n(A) = \text{pcl}_n(\text{pint}_m(A)) = \text{pint}_m(A) \cup \text{cl}(\text{int}_m(A)) \subseteq \text{cl}(\text{int}_m(A)) \cup \text{int}(\text{cl}_m(A))$.

**Theorem 3.8:** Let $A$ be a subset of a topological space $(X, \tau)$. Then the following are equivalent:

1. $A$ is a $S_nP_m$-closed set,
2. $A = \text{scl}_n(A) \cap \text{pcl}_m(A)$,
3. $A = A \cup \text{pint}_n(\text{pcl}_m(A))$,
4. $\text{pint}_n(\text{pcl}_m(A)) \subseteq A$,
5. there exists $U \in m\text{PC}(X)$ such that $\text{pint}_n(U) \subseteq A \subseteq U$,
6. $\text{pint}_n(A) = \text{pint}_n(\text{pcl}_m(A))$.

**Proposition 3.1:** If $A$ is a $S_nP_m$-open (resp. a $S_nP_m$-closed) subset of a topological space $(X, \tau)$ such that $A \subseteq B \subseteq \text{pcl}_n(A)$ (resp. $\text{pint}_n(A) \subseteq B \subseteq A$), then $B$ is $S_nP_m$-open (resp. $S_nP_m$-closed).

**4. $S_nP_m$-Continuous Functions**

**Definition 4.1:** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called $S_nP_m$-continuous if $f^{-1}(V)$ is a $S_nP_m$-open set of $X$, for each $V \in \sigma$.

**Theorem 4.1:** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a functions. Then the following statements are equivalent:

1. $f$ is $S_nP_m$-continuous,
2. For each $x \in X$ and $V \in \sigma$ containing $f(x)$, there exists $U \in S_nP_m\text{O}(X)$ containing $x$ such that $f(U) \subseteq V$,
3. The inverse image of each closed set in $Y$ is $S_nP_m$-closed in $X$. 

\[ (4) \text{int}(\text{cl}_m(f^{-1}(B))) \cap \text{cl}(\text{int}_m(f^{-1}(B))) \subseteq f^{-1}(\text{cl}(B)), \text{for each } B \subseteq Y. \]

\[ (5) f^{-1}(\text{int}(B)) \subseteq \text{cl}(\text{int}_m(f^{-1}(B))) \cup \text{int}(\text{cl}_m(f^{-1}(B))), \text{for each } B \subseteq Y, \]

\[ (6) f^{-1}(\text{int}(B)) \subseteq \text{pcl}_m(\text{pint}_m(f^{-1}(B))), \text{for each } B \subseteq Y, \]

\[ (7) \text{pint}_m(\text{pcl}_m(f^{-1}(B))) \subseteq f^{-1}(\text{cl}(B)), \text{for each } B \subseteq Y, \]

\[ (8) S_n P_m \cdot \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)), \text{for each } B \subseteq Y, \]

\[ (9) f(S_n P_m \cdot \text{cl}(A)) \subseteq \text{cl}(f(A)), \text{for each } A \subseteq X. \]

**Proof:** (1) \(\rightarrow\) (2), (2) \(\rightarrow\) (3). Obvious,

(3) \(\rightarrow\) (4). Let \(B \subseteq Y\), then by (3), \(f^{-1}(\text{cl}(B))\) is \(S_n P_m\)-closed, this implies that

\[ f^{-1}(\text{cl}(B)) \supseteq \text{int}_m(f^{-1}(\text{cl}(B))) \cap \text{cl}(\text{int}_m(f^{-1}(\text{cl}(B)))) \supseteq \text{int}_m(f^{-1}(B)) \cap \text{cl}(\text{int}_m(f^{-1}(B))), \]

(4) \(\rightarrow\) (5). By replacing \(Y \setminus B\) instead of \(B\) in (4), we have

\[ \text{int}_m(f^{-1}(Y \setminus B)) \cap \text{cl}(\text{int}_m(f^{-1}(Y \setminus B))) \subseteq f^{-1}(\text{cl}(Y \setminus B)) \text{ and therefore} \]

\[ f^{-1}(\text{int}(B)) \subseteq \text{cl}(\text{int}_m(f^{-1}(B))) \cup \text{int}(\text{cl}_m(f^{-1}(B))), \]

(5) \(\rightarrow\) (6). Let \(B \subseteq Y\). Then by (5) and Lemma 3.1, we have

\[ f^{-1}(\text{int}(B)) \subseteq f^{-1}(B) \cap (\text{cl}(\text{int}_m(f^{-1}(B))) \cup \text{int}(\text{cl}_m(f^{-1}(B)))) \]

\[ \subseteq (f^{-1}(B) \cap \text{int}(\text{cl}_m(f^{-1}(B)))) \cup \text{int}(\text{cl}_m(f^{-1}(B))) \]

\[ = \text{pcl}_m(f^{-1}(B)) \cup \text{int}_m(f^{-1}(B))) = \text{pint}_m(\text{pcl}_m(f^{-1}(B))), \]

(6) \(\rightarrow\) (7). Obvious,

(7) \(\rightarrow\) (1). Let \(B \subseteq Y\) be closed. Then \(f^{-1}(B) = f^{-1}(\text{cl}(B))\). By hypothesis

\[ \text{pint}_m(\text{pcl}_m(f^{-1}(B))) \subseteq f^{-1}(\text{cl}(B)) = f^{-1}(B). \text{Thus by Theorem 3.8, we have } f^{-1}(B) \text{ is } S_n P_m\text{-closed.} \]

By (1) \(\leftrightarrow\) (3) we have \(f\) is \(S_n P_m\)-continuous,

(1) \(\rightarrow\) (8). Let \(B \subseteq Y, f^{-1}(\text{cl}(B))\) is \(S_n P_m\)-closed in \(X\). Then \(S_n P_m \cdot \text{cl}(f^{-1}(B)) \subseteq S_n P_m \cdot \text{cl}(f^{-1}(\text{cl}(B))) = f^{-1}(\text{cl}(B)), \)
(8) \(\rightarrow(9).\) Let \(A \subseteq X.\) Then \(f(A) \subseteq Y,\) by (2), \(f^{-1}(cl(f(A))) \supseteq S_nP_{m^{-1}}cl(f^{-1}(f(A))) \supseteq S_nP_{m^{-1}}cl(A),\) therefore, \(cl(f(A)) \supseteq f^{-1}(cl(f(A))) \supseteq f_\sigma(S_nP_{m^{-1}}cl(A)),\)

(9) \(\rightarrow(1).\) Let \(W \subseteq Y\) be an open set. Then, \(F = Y \setminus W\) is closed in \(X\) and

\[ f^{-1}(F) = X \setminus f^{-1}(W). \]

Hence, by (3), \(f^{-1}(S_nP_{m^{-1}}cl(f^{-1}(F))) \subseteq cl(f(f^{-1}(F))) \subseteq cl(F) = F\) thus, \(S_nP_{m^{-1}}cl(f^{-1}(F)) \subseteq f^{-1}(F),\) So, \(f^{-1}(F) = X \setminus f^{-1}(W) \in S_nP_mC(X)\) and therefore \(f^{-1}(W) \in S_nP_mO(X)\)

Therefore, \(f\) is \(S_nP_m\)-continuous,

**Definition 4.2:** \(S_nP_{m^{-1}}Bd(A) = A \setminus S_nP_{m^{-1}}int(A)\) is said to be \(S_nP_m\)-border of \(A.\)

**Theorem 4.2:** Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a functions. Then the following statements are equivalent:

(1) \(f\) is \(S_nP_m\)-continuous,

(2) \(f^{-1}(int(B)) \subseteq S_nP_{m^{-1}}int(f^{-1}(B)),\) for each \(B \subseteq Y.\)

(3) \(S_nP_{m^{-1}}Bd(f^{-1}(B)) \subseteq f^{-1}(Bd(B)),\) for each \(B \subseteq Y,\)

**Proof:** (1) \(\rightarrow(2).\) Let \(B \subseteq Y.\) Then \(f^{-1}(int(B))\) is a \(S_nP_m\)-open set in \(X.\) Thus, \(f^{-1}(int(B)) = S_nP_{m^{-1}}int(f^{-1}(int(B))) \supseteq S_nP_{m^{-1}}int(f^{-1}(B)).\) So, \(f^{-1}(int(B)) \subseteq S_nP_{m^{-1}}int(f^{-1}(B)),\)

(2) \(\rightarrow(1).\) Let \(U \subseteq Y\) be an open set. Then \(f^{-1}(U) = f^{-1}(int(U)) \subseteq S_nP_{m^{-1}}int(f^{-1}(U)).\) Hence, \(f^{-1}(U)\) is \(S_nP_m\)-open in \(X.\) Therefore, \(f\) is \(S_nP_m\)-continuous.

(2) \(\rightarrow(3).\) Let \(B \subseteq Y.\) Then by (2), \(f^{-1}(int(B)) \subseteq S_nP_{m^{-1}}int(f^{-1}(B))\) we have

\[ f^{-1}(B) \setminus S_nP_{m^{-1}}int(f^{-1}(B)) \subseteq f^{-1}(B) \setminus f^{-1}(int(B)) \implies S_nP_{m^{-1}}Bd(f^{-1}(B)) \subseteq f^{-1}(Bd(B)),\]

(3) \(\rightarrow(2).\) Let \(B \subseteq Y.\) Then by (3), \(S_nP_{m^{-1}}Bd(f^{-1}(B)) = f^{-1}(B) \setminus S_nP_{m^{-1}}int(f^{-1}(B)) \subseteq f^{-1}(Bd(B)) = f^{-1}(B \setminus int(B)) = f^{-1}(B) \setminus f^{-1}(int(B))\) this implies \(f^{-1}(int(B)) \subseteq S_nP_{m^{-1}}int(f^{-1}(B)).\)

**Definition 4.3:** A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is called:

(1) \(S_nP_m\)-irresolute if \(f^{-1}(V) \in S_nP_mO(X, \tau),\) for each \(V \in S_nP_mO(Y, \sigma),\)

(2) \(S_nP_m\)-open (resp. \(S_nP_m\)-closed) if the \(f(V) \in S_nP_mO(Y, \sigma)\) (resp. \(S_nP_mC(Y, \sigma)\), for each \(V\) is an open (resp. closed) set of \((X, \tau)\), image of each open (resp. closed) set of \((X, \tau)\) is \(S_nP_m\)-open (resp. \(S_nP_m\)-closed) in \((Y, \sigma)\).

(3) pre- \(S_nP_m\)-open (resp. pre- \(S_nP_m\)-closed) if \(f(V) \in S_nP_mO(Y, \sigma)\) (resp. \(S_nP_mC(Y, \sigma)\), for each \(V \in S_nP_mO(X, \tau)\) (resp. \(S_nP_mC(X, \tau)\).

**Theorem 4.3:** Let \(f: X \rightarrow Y\) and \(g: Y \rightarrow Z\) be two functions. Then the following are hold:
(1) \(g \circ f\) is \(S_nP_m\)-continuous if \(f\) is \(S_nP_m\)-continuous (resp. \(S_nP_m\)-irresolute) and \(g\) is continuous (resp. \(S_nP_m\)-continuous) functions.

(2) If \(f\) is \(S_nP_m\)-continuous surjection and \(g \circ f\) is pre- \(S_nP_m\)-open (resp. pre- \(S_nP_m\)-closed), then \(g\) is \(S_nP_m\)-open (resp. \(S_nP_m\)-closed).

(3) If \(g \circ f\) is open (resp. closed) and \(g\) is injective \(S_nP_m\)-continuous, then \(f\) is \(S_nP_m\)-open (resp. \(S_nP_m\)-closed).

**Proof:** (1), (2). Obvious.

(3) Let \(A\) be an open set of \(X\). Then, \(g(f(A))\) is open in \(Z\). Since \(g\) is an injective \(S_nP_m\)-continuous, then \(g^{-1}(g(f(A))) = f(A)\) is \(S_nP_m\)-open in \(Y\). Hence, \(f\) is a \(S_nP_m\)-open function.

### 5. Some Properties

**Definition 5.1:** Let \((X, \tau)\) be a topological space and \(A \subseteq X\). Then the \(S_nP_m\)-frontier of \(A\) (briefly, \(S_nP_m\)-Fr\((A)\)) is defined by \(S_nP_m\)-Fr\((A) = S_nP_m\)-cl\((A) \cap S_nP_m\)-cl\((X \setminus A)\).

**Theorem 5.1:** The set of all points \(x\) of \(X\) at which a function \(f: (X, \tau) \to (Y, \sigma)\) is not \(S_nP_m\)-continuous is identical with the union of the \(S_nP_m\)-frontiers of the inverse images of open sets containing \(f(x)\).

**Proof:** Necessity. Let \(x\) be a point of \(X\) at which \(f\) is not \(S_nP_m\)-continuous. Then, there is an open set \(V\) of \(Y\) containing \(f(x)\) such that \(U \cap (X \setminus f^{-1}(V)) \neq \emptyset\), for every \(U \in S_nP_mO(X)\) containing \(x\). Thus, we have \(x \in S_nP_m\)-cl\((X \setminus f^{-1}(V)) = X \setminus S_nP_m\)-int\((f^{-1}(V))\) and \(x \in f^{-1}(V)\). Therefore, we have \(x \in S_nP_m\)-Fr\((f^{-1}(V))\).

Sufficiency. Suppose that \(x \in S_nP_m\)-Fr\((f^{-1}(V))\), for some open set \(V\) containing \(f(x)\). Assume that \(f\) is \(S_nP_m\)-continuous at \(x \in X\). Then by Theorem 4.1, there exists \(U \in S_nP_mO(X)\) containing \(x\) such that \(f(U) \subseteq V\). Therefore, we have \(x \in U \subseteq f^{-1}(V)\) and hence \(x \in S_nP_m\)-int\((f^{-1}(V)) \subseteq X \setminus S_nP_m\)-Fr\((f^{-1}(V))\) which is a contradiction with the assumption. This means that \(f\) is not \(S_nP_m\)-continuous at \(x\).

**Lemma 5.1:** A subset \(S\) of a space \((X, \tau)\) is \(S_nP_m\)-open if and only if \(S \cap G \in S_nP_mO(X)\), for each \(\theta\)-open set \(G\) of \(X\).

**Lemma 5.2:** Let \(A\) and \(B\) be two subsets of a space \((X, \tau)\). If \(A \in \tau_\theta(X)\) and \(B \in S_nP_mO(X)\), then \(A \cap B \in S_nP_mO(A)\).

**Theorem 5.2:** If \(f:(X, \tau) \to (Y, \sigma)\) is a \(S_nP_m\)-continuous function and \(A\) is \(\theta\)-open in \(X\), then the restriction \(f\setminus A: (A, \tau_\theta) \to (Y, \sigma)\) is \(S_nP_m\)-continuous.

**Proof:** Let \(V\) be an open set of \(Y\). Then by hypothesis \(f^{-1}(V)\) is \(S_nP_m\)-open in \(X\). Hence by Lemma 5.2, we have \((f\setminus A)^{-1}(V) = f^{-1}(V) \cap A \in S_nP_mO(A)\). Thus, it follows that \(f\setminus A\) is \(S_nP_m\)-continuous.
Lemma 5.3: Let $A$ and $B$ be two subsets of a space $(X, \tau)$. If $A \in \tau_\theta(X)$ and $B \in S_nP_mO(A)$, then $B \in S_nP_mO(X)$.

Theorem 5.3: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function and $\{G_i : i \in I\}$ be a cover of $X$ by $\theta$-open sets of $(X, \tau)$. If $f \in \sigma(X)$, then $f$ is $S_nP_m\sigma$-continuous for each $i \in I$, then $f$ is $S_nP_m\sigma$-continuous.

Proof: Let $V$ be an open set of $(Y, \sigma)$. Then by hypothesis $f^{-1}(V) = X \cap f^{-1}(V) = \cup(G_i \cap f^{-1}(V); i \in I) = \cup((f \setminus G_i)^{-1}(V); i \in I)$. Since $f \setminus G_i$ is $S_nP_m\sigma$-continuous for each $i \in I$, then $(f \setminus G_i)^{-1}(V) \in S_nP_m\sigma(G_i)$, for each $i \in I$. By Lemma 5.3, we have $(f \setminus G_i)^{-1}(V)$ is $S_nP_m\sigma$-continuous in $X$. Therefore, $f$ is $S_nP_m\sigma$-continuous in $(X, \tau)$.

Definition 5.2: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called strongly $\theta$-continuous \cite{7} if $f^{-1}(V)$ is $\theta$-open in $X$, for each $V \in \sigma$.

Theorem 5.4: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $S_nP_m\sigma$-continuous, $g: (X, \tau) \rightarrow (Y, \sigma)$ is strongly $\theta$-continuous functions and $Y$ is a Hausdorff space, then the set $\{x \in X : f(x) = g(x)\}$ is $S_nP_m\sigma$-closed in $X$.

Proof: Let $A = \{x \in X : f(x) = g(x)\}$ and $x \notin A$. Then $f(x) \neq g(x)$. Since $Y$ is a Hausdorff space, then there exist two open sets $U$ and $V$ of $Y$ such that $f(x) \in U$, $g(x) \in V$ and $U \cap V = \emptyset$. But, $f$ is $S_nP_m\sigma$-continuous, hence there exists an $S_nP_m\sigma$-open set $G$ containing $x$ such that $f(G) \subseteq U$. Since $g$ is strongly $\theta$-continuous, then there exists a $\theta$-open set $H$ of $X$ containing $x$ such that $g(H) \subseteq V$. Now, we put $W = G \cap H$, then by Lemma 5.1, $W$ is an $S_nP_m\sigma$-open set containing $x$ and $f(W) \cap g(W) \subseteq U \cap V = \emptyset$. Therefore, $W \cap A = \emptyset$ and hence $x \notin S_nP_m\sigma\text{cl}(A)$. This shows that $A$ is $S_nP_m\sigma$-closed in $X$.

Definition 5.3: A space $X$ is said to be:

1. $S_nP_m\sigma T_0$-space if for each pair of distinct points $x$ and $y$ of $X$, there exist $S_nP_m\sigma$-open sets $U$ such that $x \in U$, $y \notin U$ or $x \notin U$ and $y \in U$.

2. $S_nP_m\sigma T_1$-space if for each pair of distinct points $x$ and $y$ of $X$, there exist two $S_nP_m\sigma$-open sets $U$ and $V$ containing $x$ and $y$ respectively such that $y \notin U$ and $x \notin V$.

3. $S_nP_m\sigma T_2$-space if for each pair of distinct points $x$ and $y$ in $X$, there exist two disjoint $S_nP_m\sigma$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$.

Theorem 5.5: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is an injective $S_nP_m\sigma$-continuous function, then $(X, \tau)$ is $S_nP_m T_i$, if $(Y, \sigma)$ is a $T_i$ space, where $i = 0, 1, 2$.

Proof: We prove that the theorem for $S_nP_m T_i$-spaces.

Let $x$, $y$ be two distinct points of $X$ and $Y$ be a $T_1$-space. Then there exist open sets $U$, $V$ in $Y$ such that $f(x) \notin U$, $f(y) \notin U$, $f(x) \notin V$ and $f(y) \notin V$. Since $f$ is $S_nP_m\sigma$-continuous, hence there exist $f^{-1}(U), f^{-1}(V)$ are two $S_nP_m\sigma$-open sets of $X$ such that $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$, $x \notin f^{-1}(V)$ and $y \notin f^{-1}(V)$. Hence, $X$ is $S_nP_m T_1$.窨
Let \( f : X \to Y \) be a function. The subset \( \{(x, f(x)) : x \in X\} \) of the product space \( X \times Y \) is called the graph of \( f \) and is denoted by \( G(f) \).

**Definition 5.4:** A function \( f : X \to Y \) has a \((S_nP_m, \tau)\)-graph if for each \((x, y) \in (X \times Y) \setminus G(f)\), there exists a \( S_nP_m \)-open set \( U \) of \( X \) containing \( x \) and an open set \( V \) of \( Y \) containing \( y \) such that \((U \times V) \cap G(f) = \emptyset\).

**Lemma 5.4:** A function \( f : X \to Y \) has a \((S_nP_m, \tau)\)-graph if and only if for each \((x, y) \in X \times Y \) such that \( y \neq f(x) \), there exists a \( S_nP_m \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y \) respectively such that \( f(U) \cap V = \emptyset \).

**Proof:** It follows directly from Definition 5.4.

**Definition 5.5:** A function \( f : X \to Y \) is called:

1. quasi \( S_nP_m \)-open if the image of every \( S_nP_m \)-open set of \( X \) is open in \( Y \),
2. contra \( S_nP_m \)-open if the image of every \( S_nP_m \)-open set of \( X \) is closed in \( Y \).

**Theorem 5.6:** If \( f : X \to Y \) is a function with the \((S_nP_m, \tau)\)-graph, then the following are hold:

1. \((X, \tau)\) is \( S_nP_m \)-\( T_1 \), if \( f \) is injective,
2. \((Y, \sigma)\) is \( T_1 \), if \( f \) is surjective,
3. \((Y, \sigma)\) is \( T_2 \), if \( f \) is surjective quasi \( S_nP_m \)-open function.

**Proof:** (1) Let \( x, y \) be two distinct points of \( X \). Then \( f(x) \neq f(y) \). Hence by Lemma 3.4, there exists a \( S_nP_m \)-open set \( U \) and an open set \( V \) containing \( x \) and \( f(y) \) respectively, such that \( f(U) \cap V = \emptyset \). Therefore, \( y \notin U \) and it follows that \( X \) is \( S_nP_m \)-\( T_1 \).

(2) Let \( y_1 \) and \( y_2 \) be two distinct points of \( Y \). Since \( f \) is surjective, then there exists \( x \in X \) such that \( f(x) = y_2 \). Hence \((x, y_1) \notin G(f)\) and by using Lemma 5.4, there exists a \( S_nP_m \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y_1 \) respectively, such that \( f(U) \cap V = \emptyset \). It follows that \( y_2 \notin V \). Therefore \( Y \) is \( T_1 \).

(3) Let \( y_1 \) and \( y_2 \) be any distinct points of \( Y \). Since \( f \) is surjective \( f(x) = y_1 \), for some \( x \in X \) and \((x, y_2) \in (X \times Y) \setminus G(f)\), then there exist a \( S_nP_m \)-open set \( U \) of \( X \) and an open set \( V \) of \( Y \) such that \((x, y_2) \in U \times V \) and \((U \times V) \cap G(f) = \emptyset \). We have \( f(U) \cap V = \emptyset \). Since \( f \) is \( S_nP_m \)-open, then \( f(U) \) is open such that \( f(x) = y_1 \in f(U) \). Therefore \( Y \) is \( T_2 \).

**Theorem 5.7:** If \( f : X \to Y \) is a contra \( S_nP_m \)-open function such that the inverse image of each point of \( Y \) is \( S_nP_m \)-closed, then \( f \) has a \((S_nP_m, \tau)\)-graph.

**Proof:** Let \((x, y) \in X \setminus G(f)\). We have \( x \notin f^{-1}(y) \). Since \( f^{-1}(y) \) is \( S_nP_m \)-closed, then there exists a \( S_nP_m \)-open set \( A \) containing \( x \) such that \( A \cap f^{-1}(y) = \emptyset \). But \( f \) is contra \( S_nP_m \)-open, then \( f(A) \) is closed. Hence there exist an open set \( B \) in \( Y \) containing \( y \) such that \( f(A) \cap B = \emptyset \). Therefore, \( f \) has a \((S_nP_m, \tau)\)-graph.
Definition 5.6: A space X is said to be \( S_n P_m \)-compact if every \( S_n P_m \)-open cover of X has a finite subcover.

Theorem 5.8: If \( f: (X, \tau) \to (Y, \sigma) \) has the \( (S_n P_m, \tau) \)-graph, then \( f(K) \) is closed in \( (Y, \sigma) \), for each subset \( K \) which is \( S_n P_m \)-compact relative to \( (X, \tau) \).

Proof: Suppose that \( y \notin f(K) \). Then \( (x, y) \notin G(f) \), for each \( x \in K \). Since \( G(f) = (S_n P_m, \tau) \)-graph, then there exist a \( S_n P_m \)-open set \( U_x \) containing \( x \) and an open set \( V_x \) of \( Y \) containing \( y \) such that \( f(U_x) \cap V_x = \emptyset \). The family \( \{U_x: x \in K\} \) is a cover of \( K \) by \( S_n P_m \)-open sets. Since \( K \) is \( S_n P_m \)-compact relative to \( (X, \tau) \), then there exists a finite subset \( K_0 \) of \( K \) such that \( K \subseteq \bigcup \{U_x: x \in K_0\} \). Let \( V = \bigcap \{V_x: x \in K_0\} \). Then \( V \) is an open set in \( Y \) containing \( y \). Therefore, we have \( f(K) \cap G \subseteq Y \left( f(U_x) \cap V_x \right) \subseteq \emptyset \). It follows that \( y \notin \text{cl}(f(K)) \). Therefore, \( f(K) \) is closed in \( (Y, \sigma) \).

Corollary 5.1: If \( f: (X, \tau) \to (Y, \sigma) \) is a \( S_n P_m \)-continuous function and \( Y \) is a Hausdorff space, then \( f(K) \) is closed in \( (Y, \sigma) \), for each subset \( K \) which is \( S_n P_m \)-compact relative to \( (X, \tau) \).

6. Summary

Some results in papers [1, 4, 5, 6, 8, 9, 10, 11, 12, 15, 17] can be considered as special results from our results in this paper. The following table shows all the sets resulting from the Definition 2.1. Similarly the following table can be configured for continuity.

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References