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Research Paper

A Coupling Method of Local Fractional Variational Iteration Method and Yang-Laplace Transform for Solving Laplace Equation on Cantor Sets

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Abstract: *A user friendly algorithm based on new local fractional variational iteration transform method (LFVITM) is proposed to solve local fractional Laplace equation on Cantor sets within local fractional derivative. Further, the same problem is solved by local fractional variational iteration method and local fractional Adomian decomposition method. The results obtained by the three methods are in agreement and hence this technique may be considered an alternative and efficient method for finding approximate solutions of both linear and nonlinear fractional differential equations. The LFVITM is a combined form of local fractional variational iteration method and Laplace transform. Illustrative examples are included to demonstrate the high accuracy and fast convergence of this new algorithm.*

Keywords: Fractional Laplace equation, Yang-Laplace transform, Local fractional variational iteration method, Cantor sets.

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1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last decade, fractional calculus has found applications in numerous seemingly diverse fields of science and engineering. Fractional differential equations are increasingly used to model problems in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes [1–19].

Several analytical and numerical techniques were successfully applied to deal with differential equations, fractional differential equations, and local fractional differential equations [20-34] ,such as the local fractional series expansion method [29], the fractional Fourier [30], the fractional Laplace transform [30], the harmonic wavelet [31,32], local fractional variational iteration method [33,34], the local fractional Adomian decomposition and function decomposition [21,22,23].

In this paper, we investigate the application of local fractional variational iteration transform method (LFVITM) for solving the Laplace equation on Cantor sets with the different fractal conditions. The structure of the paper is as follows. In Section 2, we give the concept of local fractional calculus and theory of the Yang-Laplace transform. In Section 3, we give analysis of the method used. In Section 4, we consider several illustrative examples. Finally, in Section 5, we present our conclusions.

2. Preliminaries

Definition 1 [14, 15, 21, 22]: *In fractal space, let* $f(x) \in C_\alpha(a,b)$, *local fractional derivative of* $f(x)$ *of order* α *at the point* $x = x_0$ *is given by*

$$D_x^\alpha f(x_0) = \frac{d^\alpha}{dx^\alpha} f(x)\Big|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{2.1}$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha+1)(f(x) - f(x_0))$.

Definition 2 [14, 15, 21, 22]: *A partition of the interval* $[a,b]$ *is denoted as* (t_j, t_{j+1}), $j = 0,..., N-1$, $t_0 = a$ *and* $t_N = b$ *with* $\Delta t_j = t_{j+1} - t_j$ *and* $\Delta t = \max\{\Delta t_0, \Delta t_1, ...\}$. *Local fractional integral of* $f(x)$ *in the interval* $[a,b]$ *is given by*

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha. \tag{2.2}$$

Definition 3 [14, 15, 21]: *Let* $\frac{1}{\Gamma(1+\alpha)} \int_0^\infty |f(x)|(dx)^\alpha < k < \infty$. *The Yang-Laplace transforms of* $f(x)$ *is given by*

$$L_\alpha\{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha, \ 0 < \alpha \le 1 \tag{2.3}$$

where the latter integral converges and $s^\alpha \in R^\alpha$.

Definition 4 [14, 15, 21]: *The inverse formula of the Yang-Laplace transforms of* $f(x)$ *is given by*

$$L_\alpha^{-1}\{f_s^{L,\alpha}(s)\} = f(x) = \frac{1}{(2\pi)^\alpha} \int_{\beta - i\omega}^{\beta + i\omega} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(s)(ds)^\alpha, \ 0 < \alpha \le 1 \tag{2.4}$$

where $s^\alpha = \beta^\alpha + i^\alpha \omega^\alpha$; fractal imaginary unit i^α and $\mathrm{Re}(s) = \beta > 0$.

Some Basic Properties of Local Fractional Laplace Transform (see [14, 15]).

Let $L_\alpha\{f(x)\} = f_s^{L,\alpha}(s)$ and $L_\alpha\{g(x)\} = g_s^{L,\alpha}(s)$, then we have the following formulas

$$L_\alpha\{af(x) + bg(x)\} = af_s^{L,\alpha}(s) + bg_s^{L,\alpha}(s) \tag{2.5}$$

$$L_\alpha\left\{E_\alpha(c^\alpha x^\alpha)f(x)\right\}=f_s^{L,\alpha}(s-c) \tag{2.6}$$

$$L_\alpha\left\{f^{(\alpha)}(x)\right\}=s^\alpha f_s^{L,\alpha}(s)-f(0) \tag{2.7}$$

$$L_\alpha\left\{E_\alpha(a^\alpha x^\alpha)\right\}=\frac{1}{s^\alpha-a^\alpha} \tag{2.8}$$

$$L_\alpha\left\{x^{k\alpha}\right\}=\frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}} \tag{2.9}$$

3. Analysis of Method

We consider a general nonlinear local fractional partial differential equation:

$$L_\alpha u(x,t)+R_\alpha u(x,t)+N_\alpha u(x,t)=f(x,t)\ ,t>0,\ x\in R\ ,\ 0<\alpha\le 1, \tag{3.1}$$

where $L_\alpha=\dfrac{\partial^{2\alpha}}{\partial t^{2\alpha}}$, R_α is a linear local fractional operator, N_α represents the general nonlinear local fractional operator, and $f(x,t)$ is the source term.

Applying the Yang-Laplace transform (denoted in this paper by $Ł_\alpha$) on both sides of (3.1), we get

$$Ł_\alpha\{L_\alpha u(x,t)\}+Ł_\alpha\{R_\alpha u(x,t)\}+Ł_\alpha\{N_\alpha u(x,t)\}=Ł_\alpha\{f(x,t)\} \tag{3.2}$$

Using the property of the Yang-Laplace transform, we have

$$s^{2\alpha}Ł_\alpha\{u(x,t)\}-s^\alpha u(x,0)-u_t^\alpha(x,0)=Ł_\alpha\{f(x,t)\}-Ł_\alpha\{R_\alpha u(x,t)\}-Ł_\alpha\{N_\alpha u(x,t)\} \tag{3.3}$$

Or

$$\begin{aligned}Ł_\alpha\{u(x,t)\}=&\frac{1}{s^\alpha}u(x,0)+\frac{1}{s^{2\alpha}}u_t^\alpha(x,0)+\frac{1}{s^{2\alpha}}Ł_\alpha\{f(x,t)\}-\frac{1}{s^{2\alpha}}Ł_\alpha\{R_\alpha u(x,t)\}\\&-\frac{1}{s^{2\alpha}}Ł_\alpha\{N_\alpha u(x,t)\}\end{aligned} \tag{3.4}$$

Operating with the Yang-Laplace inverse on both sides of (3.4) gives

$$u(x,t)=u(x,0)+\frac{t^\alpha}{\Gamma(1+\alpha)}u_t^\alpha(x,0)+Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\{f(x,t)-R_\alpha u(x,t)-N_\alpha u(x,t)\}\right) \tag{3.5}$$

Derivative by $\dfrac{\partial^\alpha}{\partial t^\alpha}$ both side (3.5), we have

$$u_t^\alpha(x,t)-\frac{\partial^\alpha}{\partial t^\alpha}Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\{f(x,t)-R_\alpha u(x,t)-N_\alpha u(x,t)\}\right)-u_t^\alpha(x,0)=0\ . \tag{3.6}$$

By the correction function of the irrational method

$$u_{m+1}(x,t) = u_m(x,t) - {}_0I_t^{(\alpha)}\begin{pmatrix} (u_m)_\xi^\alpha(x,\xi) - \frac{\partial^\alpha}{\partial \xi^\alpha} Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}} Ł_\alpha\{f(x,\xi) - R_\alpha u_m(x,\xi) - N_\alpha u_m(x,\xi)\}\right) \\ -(u_m)_\xi^\alpha(x,0) \end{pmatrix}. \quad (3.7)$$

Finally, the solution $u(x,t)$ is given by

$$u(x,t) = \lim_{m\to\infty} u_m(x,t) \quad (3.8)$$

4. Illustrative Examples

In this section two examples for Laplace equation on Cantor sets is presented in order to demonstrate the simplicity and the efficiency of the above method.

Example 1: we consider the following Laplace equation on Cantor sets with local fractional operator

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} = 0 \quad (4.1)$$

and subject to the initial conditions

$$u(x,0) = -E_\alpha(x^\alpha)\ ,\ \frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = 0 \quad (4.2)$$

Applying the algorithm of Yang-Laplace transform on equation (4.1), we have

$$s^{2\alpha}Ł_\alpha\{u(x,t)\} - s^\alpha u(x,0) - u_t^\alpha(x,0) = -Ł_\alpha\{u_x^{2\alpha}(x,t)\}. \quad (4.3)$$

Using given initial conditions on Eq. (4.3), we have

$$s^{2\alpha}Ł_\alpha\{u(x,t)\} = -s^\alpha E_\alpha(x^\alpha) - Ł_\alpha\{u_x^{2\alpha}(x,t)\}, \quad (4.4)$$

or

$$Ł_\alpha\{u(x,t)\} = -\frac{1}{s^\alpha}E_\alpha(x^\alpha) - \frac{1}{s^{2\alpha}}Ł_\alpha\{u_x^{2\alpha}(x,t)\} \quad (4.5)$$

Then applying the inverse Laplace transform to Eq. (4.5), we get

$$u(x,t) = -E_\alpha(x^\alpha) - Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\{u_x^{2\alpha}(x,t)\}\right) \quad (4.6)$$

Derivative by $\frac{\partial^\alpha}{\partial t^\alpha}$ both sides (4.6), we have

$$u_t^\alpha(x,t) + \frac{\partial^\alpha}{\partial t^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\{u_x^{2\alpha}(x,t)\}\right)\right] = 0\ . \quad (4.7)$$

Making the correction function is given

$$u_{m+1}(x,t)=u_m(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left((u_m)_\xi^\alpha(x,\xi)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{(u_m)_x^{2\alpha}(x,\xi)\right\}\right)\right)\right)(d\xi)^\alpha \tag{4.8}$$

We can use the initial condition to select $u_0(x,t)=u(x,0)=-E_\alpha(x^\alpha)$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x,t)=-E_\alpha(x^\alpha) \tag{4.9}$$

$$u_1(x,t)=u_0(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left((u_0)_\xi^\alpha(x,\xi)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{(u_0)_x^{2\alpha}(x,\xi)\right\}\right)\right)\right)(d\xi)^\alpha$$

$$=-E_\alpha(x^\alpha)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\left\{-E_\alpha(x^\alpha)\right\}\right)\right)\right)(d\xi)^\alpha$$

$$=-E_\alpha(x^\alpha)+\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{3\alpha}}E_\alpha(x^\alpha)\right)\right)\right)(d\xi)^\alpha$$

$$=-E_\alpha(x^\alpha)\left(1-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right), \tag{4.10}$$

$$u_2(x,t)=u_1(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left((u_1)_\xi^\alpha(x,\xi)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{(u_1)_x^{2\alpha}(x,\xi)\right\}\right)\right)\right)(d\xi)^\alpha$$

$$=u_1(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\left\{-E_\alpha(x^\alpha)+\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(x^\alpha)\right\}\right)\right)\right)(d\xi)^\alpha$$

$$=u_1(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(-\frac{1}{s^{3\alpha}}E_\alpha(x^\alpha)+\frac{1}{s^{5\alpha}}E_\alpha(x^\alpha)\right)\right)\right)(d\xi)^\alpha$$

$$=-E_\alpha(x^\alpha)\left(1-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right), \tag{4.11}$$

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$$u_m(x,t)=-E_\alpha(x^\alpha)\sum_{k=0}^{m}(-1)^k\frac{t^{2\alpha k}}{\Gamma(1+2\alpha k)}. \tag{4.12}$$

Finally, the solution is

$$u(x,t)=\lim_{m\to\infty}u_m(x,t)$$

$$=-E_\alpha(x^\alpha)\sum_{k=0}^{\infty}(-1)^k\frac{t^{2\alpha k}}{\Gamma(1+2\alpha k)}$$

$$= -E_\alpha(x^\alpha)\cos_\alpha(t^\alpha)\,, \quad (4.13)$$

and its graph is shown in Figure 1.

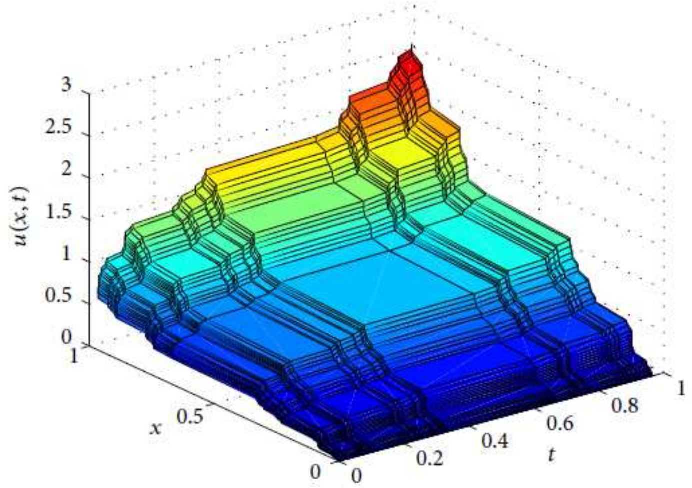

Figure 1: Exact solution for local fractional Laplace equation (4.1) with fractal dimension $\alpha = \ln 2/\ln 3$

Example 2: Consider the following Laplace equation on Cantor sets with local fractional operator

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} = 0\,, \quad (4.14)$$

and subject to the initial conditions

$$u(x,0) = 0\,,\ \frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}} = -E_\alpha(x^\alpha). \quad (4.15)$$

Applying the algorithm of Yang-Laplace transform on equation (4.14), we have

$$s^{2\alpha}Ł_\alpha\{u(x,t)\} - s^\alpha u(x,0) - u_t^\alpha(x,0) = -Ł_\alpha\left\{u_x^{2\alpha}(x,t)\right\}. \quad (4.16)$$

Using given initial conditions on Eq. (4.16), we have

$$s^{2\alpha}Ł_\alpha\{u(x,t)\} = -E_\alpha(x^\alpha) - Ł_\alpha\left\{u_x^{2\alpha}(x,t)\right\}, \quad (4.17)$$

or

$$Ł_\alpha\{u(x,t)\} = -\frac{1}{s^{2\alpha}}E_\alpha(x^\alpha) - \frac{1}{s^{2\alpha}}Ł_\alpha\left\{u_x^{2\alpha}(x,t)\right\} \quad (4.18)$$

Then applying the inverse Laplace transform to Eq. (4.18), we get

$$u(x,t) = -\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha) - Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\left\{u_x^{2\alpha}(x,t)\right\}\right) \quad (4.19)$$

Derivative by $\frac{\partial^\alpha}{\partial t^\alpha}$ both sides (4.19), we have

$$u_t^\alpha(x,t)+\frac{\partial^\alpha}{\partial t^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\left\{u_x^{2\alpha}(x,t)\right\}\right)\right]+E_\alpha(x^\alpha)=0 \tag{4.20}$$

Making the correction function is given

$$u_{m+1}(x,t)=u_m(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left((u_m)_\xi^\alpha(x,\xi)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{(u_m)_x^{2\alpha}(x,\xi)\right\}\right)\right)+E_\alpha(x^\alpha)\right)(d\xi)^\alpha \tag{4.21}$$

We can use the initial condition to select $u_0(x,t)=u(x,0)+\frac{t^\alpha}{\Gamma(1+\alpha)}u_t^\alpha(x,0)=-\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha)$. Using this selection into the correction functional gives the following successive approximations

$$u_0(x,t)=-\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha) \tag{4.22}$$

$$u_1(x,t)=u_0(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left((u_0)_\xi^\alpha(x,\xi)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{(u_0)_x^{2\alpha}(x,\xi)\right\}\right)\right)+E_\alpha(x^\alpha)\right)(d\xi)^\alpha$$

$$=-\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha)-\frac{1}{\Gamma(1+\alpha)}$$
$$\int_0^t\left(-E_\alpha(x^\alpha)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\left\{-\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha)\right\}\right)\right)+E_\alpha(x^\alpha)\right)(d\xi)^\alpha$$

$$=-\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł\left\{-\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(x^\alpha)\right\}\right)\right)\right)(d\xi)^\alpha$$

$$=-E_\alpha(x^\alpha)\left(\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right), \tag{4.23}$$

$$u_2(x,t)=u_1(x,t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left((u_1)_\xi^\alpha(x,\xi)+\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{(u_1)_x^{2\alpha}(x,\xi)\right\}\right)\right)+E_\alpha(x^\alpha)\right)(d\xi)^\alpha$$

$$=-E_\alpha(x^\alpha)\left(\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right) \tag{4.24}$$

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$$u_m(x,t)=-E_\alpha(x^\alpha)\sum_{k=0}^m(-1)^k\frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}\ . \tag{4.25}$$

Finally, the solution is

$$u(x,t)\ =\lim_{m\to\infty}u_m(x,t)$$

$$= -E_\alpha(x^\alpha)\sum_{k=0}^{\infty}(-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}$$
$$= -E_\alpha(x^\alpha)\sin_\alpha(t^\alpha), \qquad (4.26)$$

and its graph is shown in Figure 2.

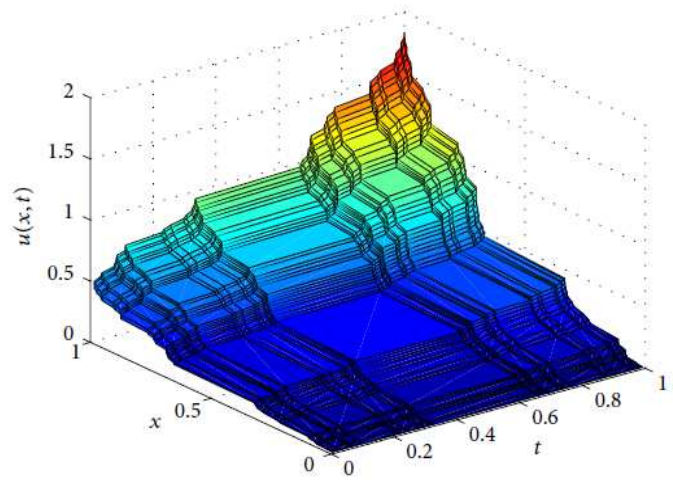

Figure 2: Exact solution for local fractional Laplace equation (4.14) with fractal dimension $\alpha = \ln 2/\ln 3$

5. Conclusions

A local fractional continuous solution to local fractional Laplace equation was developed is obtained by a new approach coupling process the local fractional variational iteration method and the Laplace transform. It may be concluded that the LFVITM is very powerful and efficient in finding the analytical solutions for a wide class of initial value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods with high accuracy of the numerical result and will considerably benefit mathematicians and scientists working in the field of partial differential equations.

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