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Research Paper

Natural Transform for Solving Fractional Systems

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Abstract: *A domain decomposition natural transform method (ADNTM) has been implemented to obtain approximate analytical solutions of fractional systems of nonlinear differential equations. The fractional derivatives are described in the Caputo sense .To illustrate the usefulness and reliability of the method, some applications are provided.*

Keywords: Natural transform, A domain decomposition natural transform method (ADNTM), Fractional systems of nonlinear differential equations.

1. Introduction

The natural transform, initially was defined by Khan and Khan [1] as N - transform, who studied their properties and applications. Later, Belgacem et al. [2, 3] defined its inverse and studied some additional fundamental properties of this integral transform and named it the natural transform. Applications of natural transform in the solution of differential and integral equations and for the distribution and Bohemians spaces can be found in [3, 4, 5, 6, 7, 8, 9, 10]. Now, we mention the following basic definitions of natural transform. With reference to [2, 11], the basic definitions of natural transform and its properties are introduced as follows:

1.1 Definition of Natural Transform

Over the set of functions

$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,\left|\; f(t)\;\right|<Me^{|t|/\tau_j},\; if\; t\in(-1)^j\times[0,\infty)\right\}$$

The natural transform of $f(t)$ is defined by

$$N[f(t)] = R(s;u) = \int_0^\infty f(ut)e^{-st}dt, u > 0, s > 0 \tag{1}$$

where $N[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables u and s are the natural transform variables.

1.2 Natural – Laplace and Sumudu Duality

If $R(s,u)$ is natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A then, $G(u)$ is Sumudu transform of function $f(t)$ in A, then:

1.2.1 Natural- Laplace Duality is:

$$N[f(t)] = R(s;u) = \frac{1}{u}\int_0^\infty f(t)e^{-\frac{st}{u}}dt = \frac{1}{u}F(\frac{s}{u}), \tag{2}$$

1.2.2 Natural-Sumudu Duality is:

$$N[f(t)] = R(s;u) = \frac{1}{s}\int_0^\infty f(\frac{ut}{s})e^{-t}dt = \frac{1}{s}G(\frac{u}{s}) \tag{3}$$

1.3 Natural Transform of nth Derivative

If $f^n(t)$ is the *n*th derivative of function $f(t)$ then, its natural transform is given by:

$$N[f^n(t)] = R_n(s,u) = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^{(k)}(0), n \ge 1 \tag{4}$$

1.4 Convolution Theorem of Natural Transform

If $F(s,u)$, $G(s,u)$ are the natural transform of respective functions $f(t)$, $g(t)$ both defined in set A then,

$$N[f * g] = uF(s,u)G(s,u) \tag{5}$$

where $f * g$ is convolution of two functions f and g.

1.5 Natural Transform of Fractional Derivative

If $N[f(t)]$ is the natural transform of the function $f(t)$, then the natural transform of fractional derivative of order α is defined as:

$$N[f^{(\alpha)}(t)] = \frac{s^\alpha}{u^\alpha}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0) \tag{6}$$

1.6 Weight Shift Property

Let the function $f(t)$ belongs to set A be multiplied with weight function $e^{\pm t}$ then,

$$N[e^{\pm t} f(t)] = \frac{s}{s \mp u} R[\frac{s}{s \mp u}] \tag{7}$$

1.7 Change of Scale Property

Let the function $f(at)$ belongs to set A, where a is non-zero constant then,

$$N[f(at)] = \frac{1}{a} R[\frac{s}{a}, u] \tag{8}$$

1.8 Natural Transform of Integrals

If $w^n(t)$ is given by $w^n(t) = \int_0^t \ldots \int_0^t f(t)(dt)^n dt$, then, the natural transform of $w^n(t)$ is given by

$$N[w^n(t)] = \frac{u^n}{s^n} R(s,u) \tag{9}$$

1.9 The Natural Transform of T-Periodic Function

The natural transform of T-periodic function $f(t) \in A$ such that $f(t+nT) = f(t), n = 0,1,2,\ldots$ is given by:

$$N[f(t)] = R(s,u) = [1 - e^{\frac{-sT}{u}}]^{-1} \frac{1}{u} \int_0^T e^{-\frac{st}{u}} f(t) dt \tag{10}$$

1.10 Multiple Shift of Natural Transform

The function $f(t)$ in set A is multiplied with shift function t^n, then

$$N[t^n f(t)] = \frac{u^n}{s^n} \frac{d}{du^n}^n u^n R(s,u) \tag{11}$$

In the present paper, we propose a new method for solving fractional systems of nonlinear differential equations called a domain decomposition natural transform method (ADNTM). It is worth mentioning that the proposed method is an elegant combination of natural transform method and a domain decomposition method. The advantage numerical solutions for fractional systems with easily computable terms.

2. Analysis of Method

To illustrate the basic idea of a domain decomposition natural transform method (ADNTM), we consider the general inhomogeneous nonlinear equation with initial conditions given below [12]:

$$LU + RU + FU = h(x,t) \tag{12}$$

Where L is the lowest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, FU represents the nonlinear terms and $h(x, t)$ is the source term**.** First, we apply natural transform on both sides of eq. (12):

$$N[LU] + N[RU] + N[FU] = N[h(x,t)] \tag{13}$$

Using the differential property of natural transform and initial conditions, we get:

$$N[h(x,t)] = \frac{s^n}{u^n} N[U(x,t)] - \frac{s^{n-1}}{u^n} U(x,0) - \frac{s^{n-2}}{u^{n-1}} U^{\backslash}(x,0) - \ldots - \frac{s}{u} U^{(n-1)}(x,0) + N[RU] + N[FU]$$

By arrangement we have:

$$N[U(x,t)] = \frac{1}{s} U(x,0) + \frac{u}{s^2} U^{\backslash}(x,0) + \ldots + \frac{u^{(n-1)}}{s^{(n-1)}} U^{(n-1)}(x,0) \\ - \frac{u^n}{s^n} N[RU] - \frac{u^n}{s^n} N[FU] + \frac{u^n}{s^n} N[h(x,t)] \quad (14)$$

The second step in natural decomposition method is that we represent solution as an infinite series:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \quad (15)$$

and the nonlinear term can be decomposed as:

$$FU(x,t) = \sum_{n=0}^{\infty} A_n \quad (16)$$

Where A_n are a domain polynomial of U_0, U_1, $U_2 \ldots\ldots U_n$ and it can be calculated by formula:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N \sum_{n=0}^{\infty} \lambda^n U_n]_{\lambda=0}$$

Substitution of (15) and (16) into (14) yields:

$$N[\sum_{n=0}^{\infty} U_n(x,t)] = \frac{1}{s} U(x,0) + \frac{u}{s^2} U^{\backslash}(x,0) + \ldots + \frac{u^{(n-1)}}{s^{(n-1)}} U^{(n-1)}(x,0) \\ - \frac{u^n}{s^n} N[RU] - \frac{u^n}{s^n} N[\sum_{n=0}^{\infty} A_n] + \frac{u^n}{s^n} N[h(x,t)] \quad (17)$$

On comparing both sides of (17) and using standard ADM we have:

$$N[\sum_{n=0}^{\infty} U_0(x,t)] = \frac{1}{s} U(x,0) + \frac{u}{s^2} U^{\backslash}(x,0) + \ldots + \frac{u^{(n-1)}}{s^{(n-1)}} U^{(n-1)}(x,0) + \frac{u^n}{s^n} N[h(x,t)] = Y(x,u)$$

Then it follows that:

$$\left.\begin{aligned} N[U_1(x,t)] &= -\frac{u^n}{s^n} N[RU_0(x,t)] - \frac{u^n}{s^n} N[A_0], \\ N[U_2(x,t)] &= -\frac{u^n}{s^n} N[RU_1(x,t)] - \frac{u^n}{s^n} u^n N[A_1]. \end{aligned}\right\} \quad (18)$$

In more general, we have:

$$N[U_{n+1}(x,t)] = -\frac{u^n}{s^n} N[RU_n(x,t)] - \frac{u^n}{s^n} N[A_n], n \geq 0 \tag{19}$$

On applying the inverse natural transform to (17) and (18), we get:

$$\left. \begin{array}{l} U_0(x,t) = K(x,t), \\ U_{n+1}(x,t) = -N^{-1}[\frac{u^n}{s^n} N[RU_n(x,t)] + \frac{u^n}{s^n} N[A_n]], n \geq 0 \end{array} \right\} \tag{20}$$

Where $K(x, t)$ represents the term that is arising from source term and prescribed initial conductions. On using the inverse natural transform to $h(x, t)$ and using the given conditions we get:

$$\Psi = \Phi + N^{-1}[h(x,t)]$$

where the functions ψ, obtained from a term by using the initial condition is given by:

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + ...+ \Psi_n ,$$

The terms ψ_0, ψ_1, ψ_2, ……., ψ_n appears while applying the inverse natural transform on the source term $h(x, t)$ and using the given conditions. We define:

$$U_0 = \Psi_k + ...+ \Psi_{k+r}$$

Where k = 0, 1, 2, 3, …, n, r = 0, 1, 2, …, n-k. Then we verify that U_0 satisfies the original equation.

3. Applications

3.1 Application

Consider the following nonlinear system [13]:

$$\left. \begin{array}{l} D_t^{\alpha} U + V_x W_y - V_y W_x = -U \\ D_t^{\alpha} V + U_x W_y + U_y W_x = V \\ D_t^{\alpha} w + U_x V_y + U_y V_x = W \end{array} \right\}, \tag{21}$$

with initial conditions:

$$U(x,y,0) = e^{x+y}, V(x,y,0) = e^{x-y}, W(x,y,0) = e^{-x+y}$$

Applying natural transform of eq. (21) and using the initial conditions, we obtain:

$$\left. \begin{array}{l} N[U(x,y,t)] = \frac{1}{s} e^{x+y} - \frac{u^{\alpha}}{s^{\alpha}} N[V_x W_y - V_y W_x + U] \\ N[V(x,y,t)] = \frac{1}{s} e^{x-y} - \frac{u^{\alpha}}{s^{\alpha}} N[U_x W_y + U_y W_x - V] \\ N[W(x,y,t)] = \frac{1}{s} e^{-x+y} - \frac{u^{\alpha}}{s^{\alpha}} N[U_x V_y + U_y V_x - W] \end{array} \right\}, \tag{22}$$

By using Eq. (15), Eq. (16) and using standard ADM, we could be able to calculate some of the terms of the series as follows:

$$U_0 = e^{x+y}, V_0 = e^{x-y}, W_0 = e^{-x+y}, \tag{23}$$

$$U_1 = -e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, V_1 = e^{x-y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, W_1 = e^{-x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{24}$$

$$U_2 = e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, V_2 = e^{x-y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, W_2 = e^{-x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{25}$$

$$U_3 = -e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, V_3 = e^{x-y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, W_3 = e^{-x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \ldots \tag{26}$$

According to ADM we have

$$U(x,t) = e^{x+y}\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(\alpha+1)} = e^{x+y}E_{\alpha,1}(-t^{\alpha}),\ V(x,t) = e^{x-y}\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha+1)} = e^{x-y}E_{\alpha,1}(t^{\alpha}),$$

$$W(x,t) = e^{-x+y}\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha+1)} = e^{-x+y}E_{\alpha,1}(t^{\alpha}).$$

Where

$$\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha+1)} = E_{\alpha,1}(t^{\alpha}),\ \sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha+1)} = E_{\alpha,1}(-t^{\alpha})$$

are Mittag-Leffer functions.

Hence, $W(x,t) = e^{-x+y+t^{\alpha}}. \quad U(x,t) = e^{x+y-t^{\alpha}},\ V(x,t) = e^{x-y+t^{\alpha}},$ (27)

At special case $\alpha \to 1$ we obtain [See figures (1, 2, 3)]:

$$U(x,y,t) = e^{x+y-t}, V(x,y,t) = e^{x-y+t}, W(x,y,t) = e^{-x+y+t} \tag{28}$$

which is the exact solution obtained by HAM[13].

3.2 Application

Consider fractional Broer-Kaup (BK) system of equations [14]:

$$\left.\begin{aligned} D_t^{\alpha}U + UU_x + V_x = 0 \\ D_t^{\alpha}V + U_x + (UV)_x + U_{xxx} = 0 \end{aligned}\right\}, \tag{29}$$

with initial condition

$$U(x,0)=1+2\tanh(x), V(x,o)=1-2\tanh^2(x)$$

As shown in the previous application, by applying ADNTM, we obtain:

$$U_0=1+2\tanh(x), V_0=1-2\tanh^2(x),$$

$$U_1=-2\frac{t^\alpha}{\Gamma(\alpha+1)}\sec h^2(x),\ V_1=4\frac{t^\alpha}{\Gamma(\alpha+1)}\sec h^2(x)\tanh(x),$$

$$U_2=-4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sec h^2(x)\tanh(x),\ V_2=4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}[\cosh(2x)-2]\sec h^4(x),$$

$$U_3=-4\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}[\cosh(2x)-2]\sec h^4(x),\dots V_3=8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}[\cosh(2x)-5]\sec h^4(x)\tanh(x),\dots$$

According to ADM we have

$$U(x,t)=1+2\tanh(x)-2\frac{t^\alpha}{\Gamma(\alpha+1)}\sec h^2(x)-4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sec h^2(x)\tanh(x)$$
$$-4\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}[\cosh(2x)-2]\sec h^4(x)+\dots, \tag{35}$$

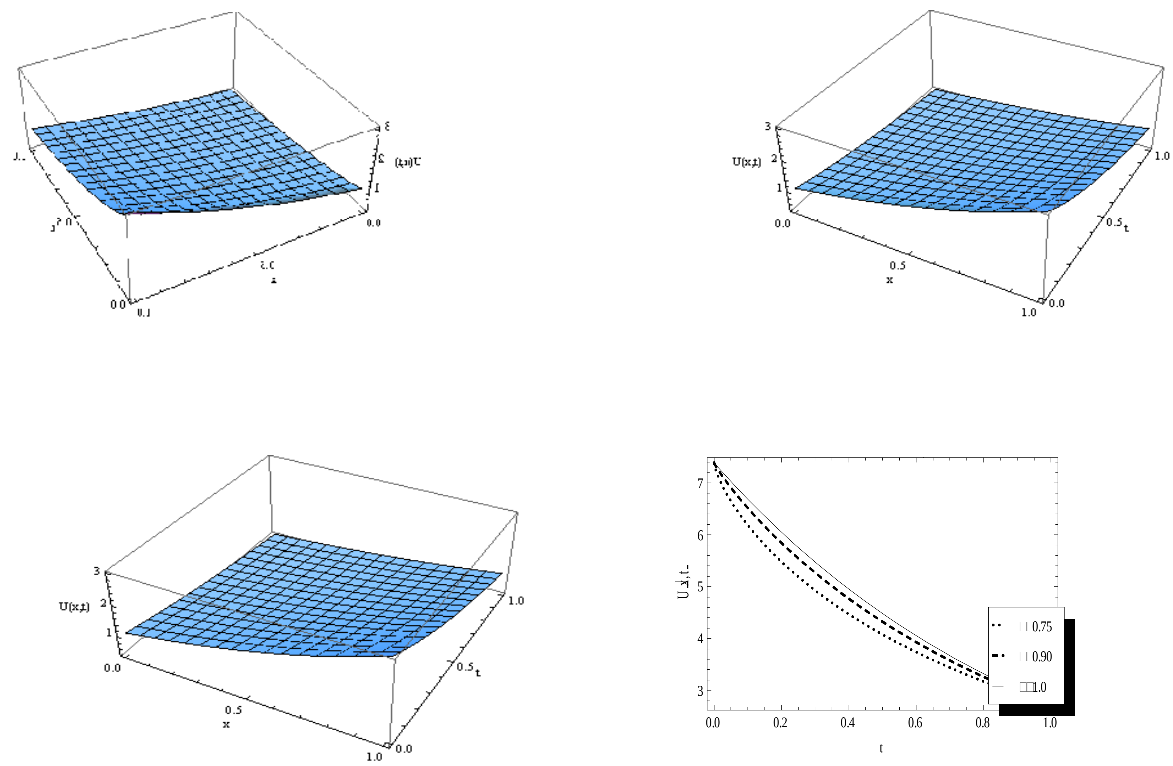
$$V(x,t)=1-2\tanh^2(x)+4\frac{t^\alpha}{\Gamma(\alpha+1)}\sec h^2(x)\tanh(x)+4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}[\cosh(2x)-2]\sec h^4(x)$$
$$+8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}[\cosh(2x)-5]\sec h^4(x)\tanh(x)+\dots \tag{36}$$

At special case $\alpha \to 1$ we obtain [See figures (4, 5)]:

$$U(x,t)=1+2\tanh(t-x), V(x,t)=1-2\tanh^2(t-x)$$

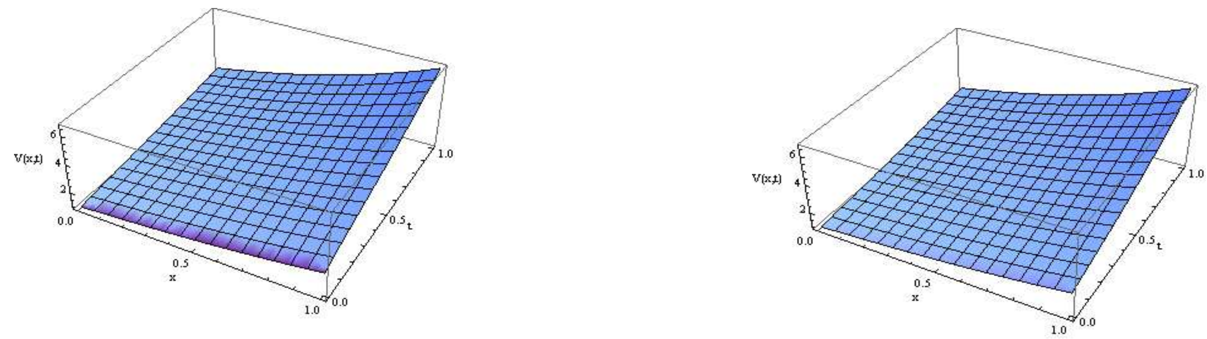
which is the exact solution obtained by [15].

4. Figures

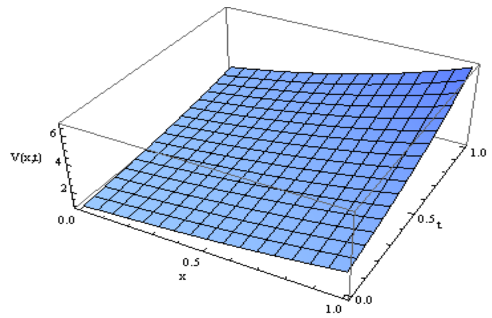


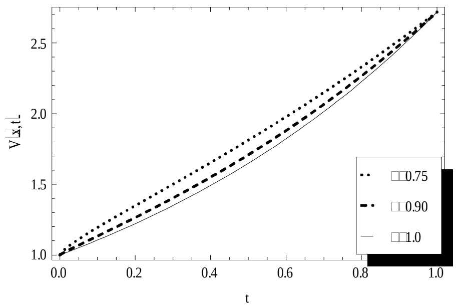
(c) **(d)**

Figure 1: The surface plot of the solution $U(x,t)$ of application 3.1 when **(a)** α = 0.75, **(b)** α =0 .90, **(c)** α =1, y=0.1 which is the exact solution, and plot2D of $U(x,t)$ versus t at x = y=1 for different values of α and comparison the results with the exact solution as shown in **(d)**



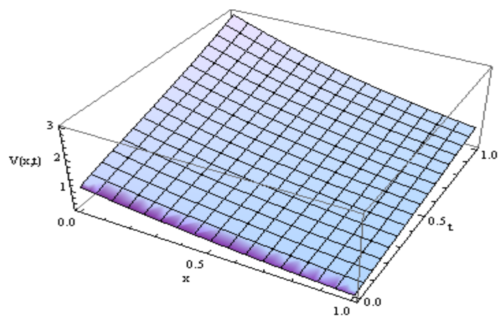
(a) **(b)**

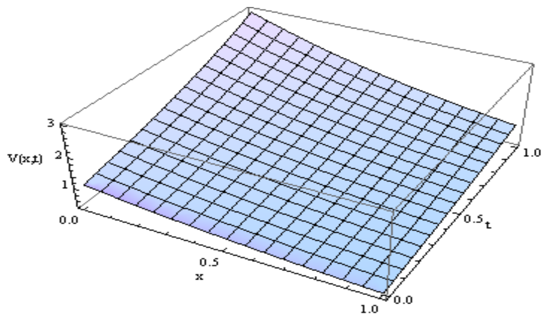




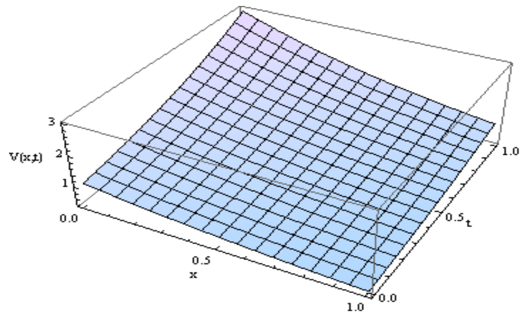
(c) **(d)**

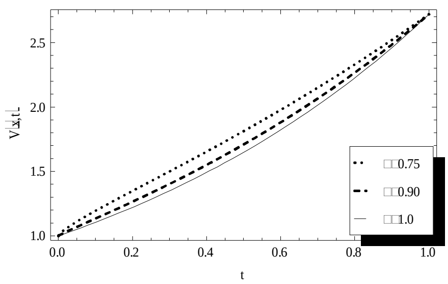
Figure 2: The surface plot of the solution $V(x,t)$ of application 3.1 when (a) α = 0.75, (b) α =0 .90, (c) α =1, y=0.1 which is the exact solution and plot2D of $V(x,t)$ versus $t$at x = y=1 for different values of$\alpha$and comparison the results with the exact solution as shown in (d)





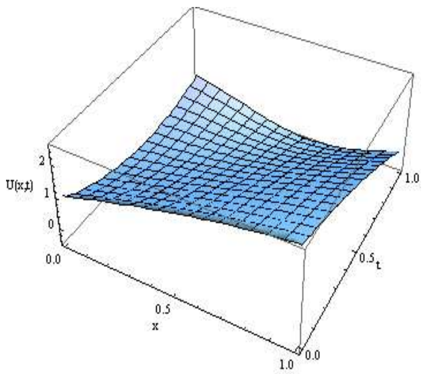
(a) **(b)**



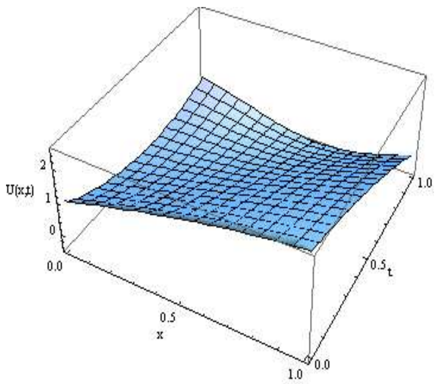


(c) **(d)**

Figure 3: The surface plot of the solution $W(x,t)$ of application 3.1 when (a) α = 0.75, (b) α =0 .90, (c) α =1, y=0.1 which is the exact solution and plot2D of $W(x,t)$ versus $t$at x = y=1 for different values of$\alpha$and comparison the results with the exact solution as shown in (d).



(a)



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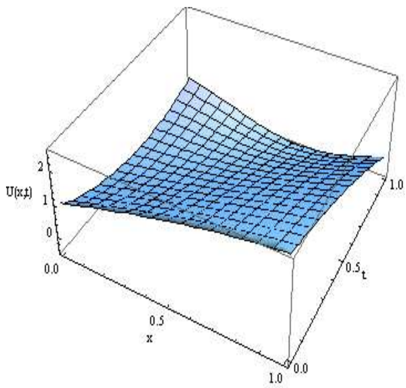
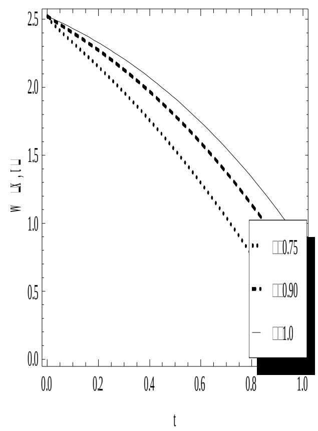
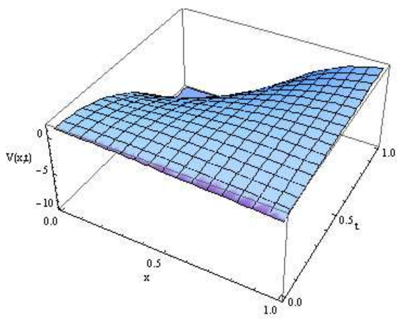


(c)

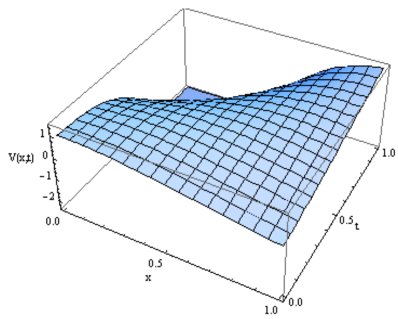


(d)

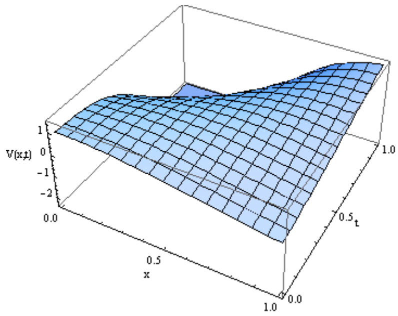
Figure 4: The surface plot of the solution $U(x,t)$ of application 3.2 when (a) α = 0.75, (b) α =0 .90, (c) α =1, which is the exact solution, and plot2D of $U(x,t)$ versus $t$at x =1 for different values of$\alpha$and comparison the results with the exact solution as shown in (d).



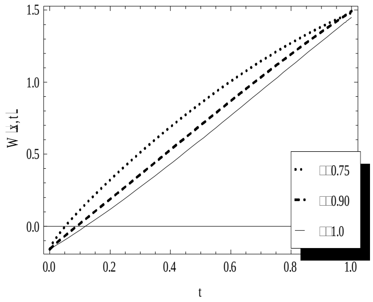
(a)



(b)



(c)



(d)

Figure 5: The surface plot of the solution $V(x,t)$ of application 3.2 when (a) α = 0.75, (b) α =0 .90, (c) α =1, which is the exact solution, and plot 2Dof $V(x,t)$ versus t at x = 1 for different values of$\alpha$and comparison the results with the exact solution as shown in (d)

5. Conclusion

As shown in the two applications, approximate analytical solutions of fractional systems of nonlinear differential equations obtained by ADNTM were in excellent agreement with the exact solutions. Finally, generally speaking, the proposed method can be further implemented to solve other physical models in fractional calculus field with easily computable terms.

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