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Research Paper

Discrete Quartic Spline Interpolation over Uniform Mesh

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Abstract: *In this paper, we have investigated existence, uniqueness and error bounds for deficient discrete quartic spline over uniform mesh.*

Keywords: Deficient, Discrete, Quartic spline, Interpolation, Error bounds.

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1. Introduction:

Discrete spline has been introduced by Mangasarian and Schumaker [4, 5] in connection with certain studied of minimization problem involving differences. Totally positivity of discrete spline collcation matrix is obtained by Rong Qing Jia [6]. Mangasarian and Schumaker [4] (See also Malcolm [3]) introduced Best summation formula for discrete spline. For some constrictive aspect of discrete spline reference may be made to Schumaker [2], Rana [8], Rana and Dubey [7]. In the present paper, we have obtained existence, uniqueness and error bounds of deficient discrete quartic spline interpolation matching the given function at two interior points, boundary points and first difference at interior points.

Let us consider a mesh P on $[0,1]$ defined by

$$P: 0 = x_0 < x_1 < .... < x_n = 1 \text{ Such that } x_i - x_{i-1} = \frac{1}{n} \text{ for i = 1, 2...., n.}$$

For a given h > 0, suppose a real continuous function $s(x,h)$ defined over [0, 1] and its restriction s_i on $[x_{i-1}, x_i]$ is a polynomial of degree 4 or less for i = 1, 2,...., n. The class S(m, r, p, h) of deficient discrete splines of degree m, with deficiency r is the set of all continuous functions $s(x,h)$ such that i=1,2,...., n-1, the restriction s_i of $s(x,h)$ on $[x_{i-1}, x_i]$ is a polynomial of degree m or less and satisfies

$$D_n^{(j)} s_i(x,h) = D_n^{(j)} s_{i+1}(x,h) \quad \text{j = 0, 1,...., m-r-1.}$$

Where the difference operator $D_n^{(i)}$ *f* or a function *f* is defined by

$$D_n^{(0)} f(x) = f(x),$$
$$D_n^{(1)} f(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

2. Existence and Uniqueness

Let the class S* (4, 1, P, h) of deficient discrete quartic spline of deficiency one over *P* satisfy the boundary conditions.

$$s(x_0,h) = f(x_0,h)$$
$$s(x_n,h) = f(x_n,h) \tag{2.1}$$

Problem 2.1: Given h>0 for what restriction on *P* Does there exist a unique $s(x,h) \in S^*(4,1,P,h)$ which satisfies following interpolating conditions.

$$s(\alpha_i) = f(\alpha_i) \tag{2.2}$$
$$s(\beta_i) = f(\beta_i) \tag{2.3}$$
$$D_n^{(1)} s(\gamma_i) = D_n^{(1)} f(\gamma_i) \tag{2.4}$$

Where

$$\alpha_i = x_{i-1} + \frac{1}{3n} = \gamma_i$$
$$\beta_i = x_{i-1} + \frac{1}{2n} \qquad \text{i = 1,2,.....n}$$

Let P(z) be a quartic polynomial on [0, 1]. Then we can show that

$$P(z) = P\left(\frac{1}{3}\right).E_1(z) + P\left(\frac{1}{2}\right).E_2(z) + D_h^{(1)} P\left(\frac{1}{3}\right).E_3(z)$$

$$+P(0).E_4(z) + P(1).E_5(z) \tag{2.5}$$

Where

$$E_1(z) = \frac{Z\left[-G\left(\frac{1}{6},\frac{3}{2}\right) + G\left(2,\frac{-9}{2}\right)z + G\left(\frac{-29}{6},24\right)z^2 + G(3,-18)z^3\right]}{G\left(\frac{2}{81},\frac{-1}{3}\right)}$$

$$E_2(z)=\frac{z\left[G\left(\frac{32}{81},0\right)+G\left(\frac{-224}{81},\frac{16}{3}\right)z+z^2\,G\left(\frac{160}{27},\frac{-64}{3}\right)+z^3G\left(\frac{-32}{9},16\right)\right.}{G\left(\frac{2}{81},\frac{-1}{3}\right)}$$

$$E_3(z)=\frac{z\left[-\frac{1}{9}+\frac{2}{3}z-\frac{11}{9}z^2+\frac{2}{3}z^3\right]}{G\left(\frac{2}{81},\frac{-1}{3}\right)}$$

$$E_4(z)=\frac{\left[1+G\left(\frac{-2}{9},\frac{4}{3}\right)z+G\left(\frac{58}{81},\frac{1}{3}\right)z^2-G\left(\frac{26}{27},\frac{16}{3}\right)z^3+G\left(\frac{4}{9},4\right)z^4\right]}{G\left(\frac{2}{81},\frac{-1}{3}\right)}$$

$$E_5(z)=\frac{z\left[G\left(-\frac{1}{162},\frac{1}{6}\right)+G\left(\frac{8}{162},\frac{-7}{6}\right)z+G\left(\frac{-7}{54},\frac{8}{3}\right)z^2+G\left(\frac{1}{9},-2)\right)z^3\right]}{G\left(\frac{2}{81},\frac{-1}{3}\right)}$$

And G (a, b) = a + bh^2 for real a and b.

Denoting $t=n(x-x_i),0\le t\le 1,$ we can expressed (2.5) in the form of restriction $s_i(x,h)$ of the deficient discrete quartic spline $s(x,h)$ on $[x_i,x_{i+1}]$ as follows:

$$s(x,h)=f(\alpha_i)E_1(x)+f(\beta_i)E_2(x)+D_n^{\{1\}}f(\gamma_i)E_3(x)\frac{1}{n}$$
$$+s_i(x_i)E_4(x)+s(x_{i+1})E_5(x) \qquad (2.6)$$

Observing (2.6) it may easily verify that $s_i(x,h)$ is a discrete quartic in $[x_i,x_{i+1}]$ for i=1,2,....n-1 satisfying (2.1) - (2.4), we shall apply continuity of the first difference from (1.1) for j=1, $s(x,h)$ at x_i in (2.6) to see that

$$\left\{\frac{1}{n^2}G\left(\frac{92}{81},2\right)+G\left(\frac{2326}{189},\frac{-32}{3}\right)h^2\right\}s_{i-1}+\left\{h^2G\left(\frac{23}{18},0\right)+\frac{1}{n^2}G\left(\frac{-10}{27},\frac{7}{2}\right)\right\}$$

$$s_i+\left\{\frac{1}{n^2}G\left(-\frac{1}{162},\frac{1}{6}\right)+h^2\,G\left(\frac{-7}{54},\frac{8}{3}\right)\right\}s_{i+1} \qquad (2.7)$$

$=F_i$ for i = 1, 2,.....n-1

Where $F_i=\left[\left\{\frac{1}{n^2}G\left(\frac{-37}{6},59\right)+h^2\,G\left(\frac{-569}{6},-48\right)\right\}\right.$

$$f(\alpha_{i-1}) - \left\{ \frac{1}{n^2} G\left(\frac{1}{6}, \frac{-3}{2}\right) + h^2 G\left(\frac{-29}{6}, 24\right) \right\} f(\alpha_i)$$

$$+ \left\{ \frac{-1}{n^2} G\left(\frac{128}{81}, \frac{736}{9}\right) + h^2 G\left(\frac{-224}{27}, \frac{128}{3}\right) \right\} f(\beta_{i-1})$$

$$- f(\beta_i) \left\{ \frac{1}{n^2} G\left(\frac{32}{81}, 0\right) + h^2 G\left(\frac{160}{27}, \frac{-64}{3}\right) \right\}$$

$$+ \frac{1}{n} G\left(\frac{2}{9}, \frac{13}{9}\right) D_h^{(1)} f(\gamma_{i-1}) + \frac{1}{n} D_h^{(1)} f(\gamma_i) G\left(\frac{1}{9}, \frac{11}{9}\right)]$$

Existence, uniqueness of $s(x,h)$ depend on the existence of a unique solution of set of equation (2.7). It is easy to observe that in (2.7) absolute value of the coefficient of s_i dominates over the sum of the absolute values of the coefficient of s_{i+1} and s_{i-1}.

That is $j^*\left(\frac{1}{n}, h\right) = \left[\frac{1}{n^2} G\left(\frac{65}{54}, 1\right) + h^2 G\left(\frac{808}{63}, 8\right) \right]$

which is positive. Thus the coefficient matrix of equation (2.7) is diagonally dominant and hence invertible.

Remark: In case $h \to 0$ theorem (2.1) gives corresponding result for continuous quartic spline under condition (2.1) - (2.4).

3. Error Bounds:

Now system of equation (2.7) may be written as

$$A(h).M(h) = F$$

where A(h) is coefficient matrix and $M(h) = s_i(h)$, however, as already shown in the proof of theorem (2.1) - A(h) is invertible denoting the inverse of A(h) by $A^{-1}(h)$ then

$$\| A^{-1}(h) \| \le d(h) \tag{3.1}$$

where $$d(h) = \max \left\{ j_i^*\left(\frac{1}{n}, h\right) \right\}^{-1},$$

For convenience we assume in this section that $f^{(1)}$ for $D_h^{(1)} f$, and *w(f, P)* for modules of continuity of *f*, the discrete norm of a function *f* over the interval $[0, 1]_h$ is defined by

$$\| f \| = \max | f(x) | \tag{3.2}$$

and the first difference is defined by

$$[x_0, x_1]_f = \left[\frac{f(x_0) - f(x_1)}{(x_0 - x_1)}\right] \tag{3.3}$$

We shall obtain in the following the bounds of error function $e(x) = s(x,h) - f(x)$ over the discrete interval $[0, 1]_h$.

Theorem 3.1: *Suppose s(x, h) is the discrete quartic spline interpolant of Theorem 2.1 then*

$$\| e(x) \| \le K(p,h)\, w(f, p) \tag{3.4}$$

$$\| e(x) \| \le d(h)\, K^*(P,h)\, w(f, p) \tag{3.5}$$

$$\| e^{(1)}(x) \| \le K_1(P,h)\, w(f, p) \tag{3.6}$$

Where K(P, h), K* (P, h) and $K_1(P,h)$ are some positive function of P and h.

Proof: Let $e(x_i) = s(x_i, h) - f_i$ Where $f_i = f(x_i)$ (2.7) equation can be written as

$$A(h).e(x_i) = F_i(h) - A(h)\, f_i = R_i(f) \qquad \text{(Say)} \tag{3.7}$$

We need the following lemma due to Lyche [9, 10], to estimate inequality (3.4).

Lemma 3.1: *Let* $\{a_i\}_{i=1}^{m_1}$ *and* $\{b_j\}_{j=1}^{n}$ *be given sequence of non-negative real number's such that* $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ *then for any real values function f, defined on discrete interval* $[0,1]_h$ *we have*

$$\left| \sum_{i=1}^{m} a_i\, [x_{i0}, x_{i1}, \ldots\ldots x_{ik}]_f - \sum_{j=1}^{n} b_j\, [y_{j0}, y_{j_1}, \ldots\ldots y_{j_k}]_f \right|$$

$$\ge w\left(f^{(k)}, |1 - kh|\right) \frac{\sum a_i}{k!} \tag{3.8}$$

Where $x_{jk}, y_{jk} \in [0,1]_h$ for relevant value of i, j and k. We can write equation (2.7) is of the form of error function as follows:

$$\left[\left\{\frac{1}{n^2} G\left(\frac{92}{81}, 2\right) + G\left(\frac{2326}{189}, \frac{-32}{3}\right) h^2\right\} e_{i-1} + \left\{\frac{1}{n^2} G\left(\frac{-10}{27}, \frac{7}{2}\right) + G\left(\frac{23}{18}, 0\right) h^2\right\}\right.$$

$$e_i + \left\{\frac{1}{n^2} G\left(\frac{1}{162}, \frac{1}{6}\right) + G\left(\frac{-7}{54}, \frac{8}{3}\right) h^2\right\} e_{i+1} = R(f)$$

Where $R_i(f) = F_i - \left[\left\{\frac{1}{n^2} G\left(\frac{92}{81}, 2\right) + G\left(\frac{2326}{189}, \frac{-32}{3}\right) h^2\right\}\right.$

$$f_{i-1}-\left\{\frac{1}{n^2}G\left(\frac{-10}{27},\frac{7}{2}\right)+G\left(\frac{23}{18},0\right)h^2\right\}f_i$$
$$-\left\{\frac{1}{n^2}G\left(\frac{-1}{162},\frac{1}{6}\right)+G\left(\frac{7}{54},\frac{8}{3}\right)h^2\right\}f_{i+1}\Bigg] \quad (3.9)$$

Writing equation (3.9) is of the form of divided difference and using Lemma 3.1 given by Lyche [9]

$$|R_i(f)|=\left|\frac{-1}{n}[\alpha_{i-1},\beta_{i-1}]_f\left\{\frac{1}{n^2}G\left(\frac{2}{9},\frac{-7}{4}\right)+h^2G\left(\frac{43}{36},-8\right)\right\}\right.$$

$$-\frac{1}{n^3}G\left(\frac{11}{81},-\frac{1}{3}\right)[\alpha_i,x_i]_f+\frac{1}{n}[\beta_{i-1},x_i]_f\left\{\frac{1}{n^2}G\left(\frac{10}{81},-\frac{1}{12}\right)+h^2G\left(\frac{61}{108},\frac{8}{3}\right)\right\}$$

$$+\frac{1}{n}[x_{i-1},x_i]_f\left\{\frac{1}{n^2}G\left(\frac{-8}{81},-2\right)+h^2G\left(\frac{-22}{27},\frac{-32}{3}\right)\right\}$$

$$+\frac{1}{n}G\left(\frac{2}{9},\frac{13}{9}\right)[\gamma_{i-1}-h,\gamma_{i-1}+h]_f-\frac{1}{9n^3}[\gamma_i-h,\gamma_i+h]$$

$$+\frac{1}{n}\left\{\frac{1}{n^2}G\left(\frac{-16}{243},0\right)+G\left(\frac{-80}{81},\frac{64}{18}\right)h^2\right\}[\alpha_i,\beta_i]_f$$

$$-\frac{1}{n}\left\{\frac{1}{n^2}G\left(\frac{-5}{81},-\frac{1}{9}\right)+G\left(\frac{26}{81},\frac{16}{9}\right)h^2\right\}[x_i,\alpha_i]_f$$

$$+\frac{1}{n}\left\{\frac{1}{n^2}G\left(\frac{1}{243},-\frac{1}{9}\right)+G\left(\frac{7}{81},\frac{-16}{9}\right)h^2\right\}$$

$$[\alpha_i,x_{i+1}]+\frac{11}{9n}h^2[\gamma_i-h,\gamma_i+h]_f$$

$$\Rightarrow|R_i(f)|=\left|\sum_{i=1}^{5}a_i\left[x_{i_0},x_{i_1}\right]_f-\sum_{j=1}^{5}b_j\left[y_{j_0},y_{j_1}\right]_f\right| \quad (3.10)$$

$$\le w\left(f^{(1)},P\right)\left(\sum_{i=1}^{5}a_i=\sum_{j=1}^{5}b_j\right)$$

$$=\frac{1}{n}\left[\frac{1}{n^2}G\left(\frac{15}{81},\frac{-79}{36}\right)+h^2G\left(\frac{95}{324},-\frac{88}{9}\right)\right] \quad (3.11)$$

Where $a_1=\frac{1}{n}\left[\frac{1}{n^2}G\left(\frac{10}{81},-\frac{1}{12}\right)+h^2G\left(\frac{61}{108},\frac{8}{13}\right)\right],$

$$a_2 = \frac{1}{n}\left[\frac{1}{n^2} G\left(-\frac{8}{81}, -2\right) + G\left(-\frac{22}{27}, -\frac{32}{3}\right)h^2\right],$$

$$a_3 = \frac{1}{n}\left[G\left(\frac{2}{6}, \frac{13}{9}\right)\right],$$

$$a_4 = \frac{1}{n}\left[\frac{1}{n^2} G\left(-\frac{16}{243}, 0\right) + G\left(-\frac{-80}{81}, \frac{64}{18}\right)h^2\right],$$

$$a_5 = \frac{1}{n}\left[\frac{1}{n^2} G\left(\frac{1}{243}, -\frac{1}{9}\right) + G\left(\frac{7}{81}, -\frac{16}{9}\right)h^2\right]$$

$$b_1 = \frac{1}{n}\left[\frac{1}{n^2} G\left(\frac{2}{9}, \frac{-7}{4}\right) + G\left(\frac{43}{36}, -8\right)h^2\right],$$

$$b_2 = \frac{1}{n} G\left(\frac{11}{81}, \frac{-1}{3}\right),$$

$$b_3 = -\frac{1}{9n^3},$$

$$b_4 = \frac{1}{n}\left[\frac{1}{h^2} G\left(\frac{-5}{81}, \frac{-1}{9}\right) + G\left(\frac{26}{81}, \frac{16}{9}\right)h^2\right],$$

$$b_5 = -\frac{11}{9n} h^2.$$

and $x_{10} = \beta_{i-1} = y_{11}, x_{11} = x_i = x_{21} - y_{21} = y_{40}$

$$x_{20} = x_{i-1}, x_{3_0} = \gamma_{i-1} - h, x_{3_1} = \gamma_{i-1} + h$$

$$y_{10} = \alpha_{i-1}, y_{2_0} = \alpha_i = y_{4_1} = x_{4_0} = x_{5_0}$$

$$y_{30} = y_{50} = \gamma_i - h, y_{31} = \gamma_i + h = y_{51}$$

$$x_{4_1} = \beta_i, x_{5_1} = x_{i+1}$$

Now using the equation (3.1) and (3.10) in (3.7)

$$\| e(x_i) \| \le d(h) K^*(P, h)\, w\left(f^{(1)}, P\right)$$

This is the inequality (3.5) of Theorem 3.1.

Writing equation (2.7) in the form of error function as follows:

$$e(x) = e_{i-1} Q_4(t) + e_i Q_5(t) + M_i(f)$$

Where $M_i(f) = f(\alpha_i)Q_1(t) + f(\beta_i)Q_2(t)$

$$+\frac{1}{n} f^{\{1\}}(\gamma_i)Q_3(t) + f_{i-1}Q_4(t) + f_i Q_5(t) - f(x) \tag{3.12}$$

Again writing $M_i(f)$ in form of divided difference and using Lemma 3.1. We get

$$|M_i(f)| \le w\left(f^{(1)}, P\right)\sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j$$

Where $a_1 = \frac{1}{n}\left[t\,G\left(\frac{1}{36},\frac{1}{4}\right) + t^2 G\left(\frac{-1}{3},\frac{3}{4}\right) + G\left(\frac{29}{36},-4\right)t^3 + G\left(-\frac{1}{2},3\right)t^4\right],$

$$a_2 = \frac{1}{n}\left[G\left(\frac{1}{9},\frac{-2}{3}\right)t + G\left(\frac{-19}{18},\frac{-1}{6}\right)t^2 + G\left(\frac{13}{27},\frac{8}{3}\right)t^3 + G\left(\frac{5}{9},-2\right)t^4\right],$$

$$a_3 = \frac{1}{n}\left[-\frac{1}{9}t + \frac{2}{3}t^2 - \frac{11}{a}t^3 + \frac{2}{3}t^4\right],$$

$$b_1 = \frac{1}{n}\left[G\left(\frac{1}{324},\frac{1}{12}\right)t + t^2 G\left(\frac{-5}{108},\frac{59}{56}\right) - G\left(\frac{-7}{108},\frac{4}{3}\right)t^3 - G\left(\frac{1}{18},-1\right)t^4\right],$$

$$b_2 = \frac{1}{n}\left[t\,G\left(\frac{8}{27},\frac{-1}{3}\right)\right].$$

And $x_{10} = \alpha_i, x_{11} = \beta_i, x_{20} = x_{i-1}, x_{21} = \beta_i$

$$x_{30} = \gamma_i - h, x_{3_1} = \gamma_i + h, y_{10} = \beta_i, y_{1_1} = x_i$$

$$y_{2_0} = x_{i-1}, y_{2_1} = x$$

Thus $\sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j = \frac{1}{n}\left[G\left(\frac{1}{36},\frac{-5}{12}\right)t + G\left(\frac{2}{81},\frac{-7}{12}\right)\right]$

$$t^2 + G\left(\frac{7}{108},\frac{-4}{3}\right)t^3 + G\left(\frac{-5}{18},1\right)t^4] \tag{3.13}$$

From equation (3.5), (3.12) and (3.13) gives inequality (3.4) of Theorem 3.1.

We now proceed to obtain bound of $e^{\{1\{}(x)$

$$\frac{1}{n}s^{\{1\}}(x) = f(\alpha_i)E_1^{\{1\}}(t) + f(\beta_i)E_2^{\{1\}}(t) + \frac{1}{n}f^{\{1\}}(\gamma_i)E_3^{\{1\}}t$$

$$+ s(x_i)E_4^{\{1\}}(t) + s(x_{i+1})E_5^{\{1\}}(t) \tag{3.14}$$

Now $\frac{A}{n} e^{\{1\}}(x) = e_{i-1} E_4^{\{1\}}(t) + e_i E_5^{\{1\}}(t) + U_i(f)$ (3.15)

Where $U_i(f) = f(\alpha_i) E_1^{\{1\}}(t) + f(\beta_i) E_2^{\{1\}}(t) + \frac{1}{n} f^{\{1\}}(\gamma_i) E_3^{\{1\}}(t)$

$+ f_{i-1} E_4^{\{1\}}(t) + f_i E_5^{\{1\}}(t) - \frac{A}{n} f^{\{1\}}(x)$ and $A = G\left(\frac{2}{81}, \frac{-1}{3}\right)$

By using Lemma 3.1, we get

$$|U_i(f)| \le w\left(f^{(1)}, P\right) \sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j$$

$$= \frac{1}{n}\left[-G\left(\frac{1}{324}, \frac{7}{12}\right) + G\left(\frac{2}{3}, -\frac{3}{2}\right)t + \right.$$

$$G\left(\frac{-29}{36}, 4\right)\left(3t^2 + h^2\right) + G\left(\frac{1}{2}, -3\right) 4t\left(t^2 + h^2\right)] \quad (3.16)$$

Where $a_1 = \frac{1}{n}\left[G\left(\frac{1}{9}, \frac{-2}{3}\right) - G\left(\frac{58}{81}, \frac{1}{3}\right)t + \left(3t^2 + h^2\right)G\left(\frac{157}{27}, \frac{8}{3}\right) - 4G\left(\frac{2}{9}, 2\right)t\left(t^2 + h^2\right)\right],$

$$a_2 = \frac{1}{n}\left[G\left(-\frac{1}{324}, \frac{1}{12}\right) + tG\left(\frac{4}{81}, \frac{7}{6}\right) + \left(3t^2 + h^2\right)G\left(\frac{-23}{108}, \frac{4}{3}\right)\right.$$

$$\left. 4t\left(t^2 + h^2\right)G\left(\frac{1}{18}, -1\right)\right],$$

$$a_3 = \frac{1}{n}\left[-\frac{1}{9} + \frac{4}{3}t - \frac{11}{9}\left(3t^2 + h^2\right) + \frac{8}{3}t\left(t^2 + h^2\right)\right],$$

$$b_1 = \frac{1}{n}\left[G\left(\frac{1}{-36}, -\frac{1}{4}\right) + G\left(\frac{2}{3}, \frac{-3}{2}\right)t + G\left(\frac{-29}{36}, \frac{12}{3}\right)\left(3t^2 + h^2\right)\right.$$

$$\left. + 4G\left(\frac{1}{2}, -3\right)t\left(t^2 + h^2\right)\right],$$

$$b_2 = \frac{1}{n} G\left(\frac{2}{81}, \frac{-1}{3}\right).$$

And $x_{10} = x_i, x_{11} = \beta_i = x_{21}, x_{22} = x_{i+1}$

$$x_{30} = \gamma_i - h, x_{3_1} = \gamma_i + h, y_{1_0} = \alpha_i, y_{1_1} = \beta_i$$

$$y_{20} = x - h, y_{21} = x + h.$$

By using equation (3.5), (3.16) in (3.15), we get inequality (3.6) of theorem 3.1.

Conclusion

We have investigated existence, uniqueness and error bounds for deficient discrete quartic spline over uniform mesh.

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