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Research Paper

A Self-Starting First Order Initial Value Problems Solver

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Abstract: *This paper focuses on the construction of continuous approximation scheme for the solution of first order initial value problems in ordinary differential equations. We exploit here the elegant properties of the Chebyshev polynomials and derive from the continuous scheme, an implicit hybrid block method through some selected points. The self-starting method was implemented on three test problems to show the accuracy and effectiveness of the method.*

Keywords: Collocation, Continuous schemes, Chebyshev polynomials, Interpolate.

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1. Introduction

The desirability of deriving a continuous scheme for solving first order and higher order ordinary differential equations cannot be over emphasized. This is as a result of the need to increase the effectiveness and efficiency of multistep methods in solving differential equations. Over the years, techniques for the derivation of linear multistep methods (LMMs) for the numerical solution of the initial values problems (IVPs) in first order ordinary differential equation of the form:

$$y'(x) = f\left(x,\ \mathrm{y}(\mathrm{x})\right),\ a \le x \le b < +\infty \qquad (1.1a)$$

$$y(a) = y_0 \qquad (1.1b)$$

have been reported in the literature, see ([1-4], [6]) and these includes, among others, collocation, interpolation, integration of interpolation polynomials. Various types of basis functions such as the Legendre polynomials $P_n(x)$, the monomials x^r, the Canonical polynomials $(Q_r(x), r \ge 0)$ of the

Lanczos Tau method in a perturbed collocation approach have been employed for this purpose, see ([7-12],).

In our previous paper, see ([1], [5]), the choice of the Chebyshev polynomials have been considered with great success. This suggests the drive for the present paper on the hybrid block method where Chebyshev polynomial is considered as the trial function and the hybrid block method which simultaneously generates solutions of (1) without the need for any predictor. In the next section, the development of the method is considered.

2. Materials and Methods

Here, we first construct the continuous scheme which at a desired point, yield corresponding explicit method. We shall consider an approximation of the form

$$y(x) = \sum_{r=0}^{n} a_r T_r(x) \tag{1.2}$$

To the solution y(x) of (1.1) in the interval $[x_n, x_{n+1}]$ where n=s+k-1, s is the interpolation point and k is the point at which collocation is performed.

We interpolate (1.1) at one point x = x_n and collocate the derivative of (1.1) at points $x = x_n, x = x_{n+\frac{1}{3}}, x = x_{n+\frac{2}{3}}$ *and* $x = x_{n+1}$ to obtain a system of equations written in the matrix form as

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & -8 & 18 & -32 \\ 0 & 2 & \frac{-8}{3} & \frac{-10}{3} & \frac{224}{27} \\ 0 & 2 & \frac{8}{3} & \frac{-10}{3} & \frac{224}{27} \\ 0 & 2 & 8 & 18 & 32 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_n \\ hf_n \\ hf_{n+\frac{1}{3}} \\ hf_{n+\frac{2}{3}} \\ hf_{n+1} \end{pmatrix} \tag{1.3}$$

Solving equation (1.3) by Gaussian elimination method, the values of the unknown parameters a_j, j = 0, 1, 2, 3 are obtained as

$$\begin{aligned}
a_0 &= y_n + \frac{h}{1024}(101 f_n + 273 f_{n+\frac{1}{3}} + 111 f_{n+\frac{2}{3}} + 27 f_{n+1}) \\
a_1 &= \frac{h}{128}(5 f_n + 27 f_{n+\frac{1}{3}} + 27 f_{n+\frac{2}{3}} + 5 f_{n+1}) \\
a_2 &= \frac{h}{256}(-7 f_n - 27 f_{n+\frac{1}{3}} + 27 f_{n+\frac{2}{3}} + 7 f_{n+1}) \\
a_3 &= \frac{h}{128}(3 f_n - 3 f_{n+\frac{1}{3}} - 3 f_{n+\frac{2}{3}} + 3 f_{n+1}) \\
a_4 &= \frac{h}{1024}(-9 f_n + 27 f_{n+\frac{1}{3}} - 27 f_{n+\frac{2}{3}} - 27 f_{n+1})
\end{aligned} \tag{1.4}$$

Substituting (1.4) into (1.2) yields the continuous implicit hybrid scheme

$$y(x) = \alpha_0(t)y_n + h(\beta_0(t)f_n + \beta_{\frac{1}{3}}(t)f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}(t)f_{n+\frac{2}{3}} + \beta_1(t)f_{n+1} \quad (1.5)$$

Equation (1.5) yields the parameters $\alpha(t)$ and $\beta(t)$ as the following continuous functions:

$$\begin{aligned}
&\alpha_0(t) = 1\\
&\beta_0(t) = -\frac{9}{128}t^4 + \frac{3}{32}t^3 + \frac{1}{64}t^2 - \frac{1}{32}t + \frac{15}{128}\\
&\beta_{\frac{1}{3}}(t) = \frac{27}{128}t^4 - \frac{3}{32}t^3 - \frac{27}{64}t^2 + \frac{9}{32}t + \frac{51}{128}\\
&\beta_{\frac{2}{3}}(t) = -\frac{27}{128}t^4 - \frac{3}{32}t^3 + \frac{27}{64}t^2 + \frac{9}{32}t - \frac{3}{128}\\
&\beta_1(t) = \frac{9}{128}t^4 + \frac{3}{32}t^3 - \frac{1}{64}t^2 - \frac{1}{32}t + \frac{1}{128}
\end{aligned} \quad (1.6)$$

where $t = \dfrac{2(x - x_n) - h}{h}$.

Evaluating (1.5) at $x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}$ *and* x_{n+1} gives the explicit scheme in block form as

$$y_m = hBF(y_m) + Ey_n + hDf_n \quad (1.7)$$

where

$$y_m = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{bmatrix}, B = \begin{bmatrix} \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} \frac{9}{72} \\ \frac{1}{9} \\ \frac{1}{8} \end{bmatrix}$$

3. The Basic Properties of the Method

3.1 Order, Error Constant and Consistency of the Method

The discrete schemes in block form (1.7) derived belong to the class of Linear Multistep Method (LMM) of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} \quad (1.8)$$

Associated with (1.8) is the linear differential operator L defined by

$$L[y(x);h] = \sum_{j=0}^{k} [\alpha_j y(x + jh) - h\beta_j y'(x + jh)] \quad (1.9)$$

Expanding (1.9) by Taylor series, we have

$$L[y(x);h]=C_0y(x)+C_1hy'(x)+...+C_qh^qy^{(q)}(x)+... \qquad (1.10)$$

where the C_q are constants.

According to this definition, the LMM (1.8) is said to be of order p if $C_0 = C_1 = C_2 = . . . = C_p = 0$ and $C_{p+1} \neq 0$. The $C_{p+1} \neq 0$ is the error constant, see Lambert (1973). According to this definition, all the members of the block hybrid method of the form (1.7) have uniform order p = 4 with error constants $C_{p+1}=(-\frac{19}{174960},-\frac{1}{21870},-\frac{1}{6480})^T$.

Consistency: The LMM (1.8) is said to be consistent if it is of order $p \geq 1$, see [7]. The schemes in (1.7) satisfy this condition.

Zero-Stability: The LMM (1.8) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root of modulus one has multiplicity not greater than one (the order of the differential equation), see [7]. The schemes in (1.7) satisfy this condition.

Numerical Examples: Here, we consider the application of the derived schemes to three test problems to know the efficiency and accuracy of the method implemented as block method.

Problem 1:

$y' = 1 + y^2$, $y(0) = 0$, $h = 0.01$

Exact solution: y(x) = tanx.

Table 1: Comparison of absolute errors

X	Exact Solution	New Method (NM)	NM Error	Error in [6]
0.01	0.010000333346667	0.010000333346914	2.469864590626258e-013	9.0581653300 E – 5
0.02	0.020002667093402	0.020002667093897	4.942990461387353e-013	1.6100290660 E – 4
0.03	0.030009003241181	0.030009003241924	7.431243120858966e-013	2.1128275880 E – 4
0.04	0.040021346995515	0.040021346996508	9.933928679650705e-013	2.4144200450 E – 4
0.05	0.050041708375539	0.050041708376784	1.245635539159906e-012	2.5149362450 E – 4
0.06	0.060072103831297	0.060072103832797	1.500043145252761e-012	2.4144316870 E – 4
0.07	0.070114557872003	0.070114557873760	1.757718970374356e-012	2.1129112800 E – 4
0.08	0.080171104708073	0.080171104710092	2.019009959219886e-012	1.6102529190 E – 4
0.09	0.090243789909785	0.090243789912071	2.285477362917732e-012	9.0625090200 E – 5
0.10	0.100334672085451	0.100334672088007	2.556455047653117e-012	5.4915000000 E – 8

Problem 2:

$y' = -y$, $y(0) = 1$, $h = 0.1$

Exact solution: $y(x) = e^{-x}$

Table 2: Comparison of absolute errors

X	Exact Solution	New Method (NM)	NM Error	Error in [13]
0.1	0.904837418035960	0.904837416638035	1.397924442869680e-009	$2.0x10^{-9}$
0.2	0.818730753077982	0.818730750548193	2.529788933891553e-009	$2.0\ x10^{-9}$
0.3	0.740818220681718	0.740818217248147	3.433571316158179e-009	$1.0\ x10^{-9}$
0.4	0.670320046035639	0.670320041893208	4.142431286879855e-009	$2.0\ x10^{-9}$
0.5	0.606530659712633	0.606530655027350	4.685283605532220e-009	$1.0\ x10^{-9}$
0.6	0.548811636094027	0.548811631006723	5.087303911466279e-009	$3.0\ x10^{-9}$
0.7	0.496585303791410	0.496585298421030	5.370379696501004e-009	$2.0\ x10^{-9}$
0.8	0.449328964117222	0.449328958563712	5.553509097033782e-009	$3.0\ x10^{-9}$
0.9	0.406569659740599	0.406569654087448	5.653151224915831e-009	$3.0\ x10^{-9}$
1.0	0.367879441171442	0.367879435487906	5.683536363765285e-009	$3.0\ x10^{-9}$

Problem 3:

$y' = -8 (y - x) + 1$, $y(0) = 2$, $h = 0.1$

Exact solution: $y(x) = x + 2\ e^{(-8x)}$

Table 3: Comparison of absolute errors

X	Exact Solution	New Method (NM)	NM Error	Error in [13]
0.1	0.998657928234443	0.998609151380770	4.877685367310480e-005	$3.6x10^{-4}$
0.2	0.603793035989311	0.603749203472634	4.383251667694488e-005	$1.5x10^{-4}$
0.3	0.481435906578825	0.481406364551603	2.954202722227350e-005	$5.9x10^{-5}$
0.4	0.481524407956732	0.481506709652393	1.769830433917541e-005	$1.6x10^{-5}$
0.5	0.536631277777468	0.536621337596288	9.940181180501106e-006	$4.3x10^{-5}$
0.6	0.616459494098040	0.616454134549915	5.359548125416502e-006	$2.1x10^{-5}$
0.7	0.707395727432966	0.707392917942302	2.809490663779890e-006	$5.7x10^{-7}$
0.8	0.803323114546348	0.803321671859180	1.442687168062840e-006	$1.6x10^{-6}$
0.9	0.901493171616753	0.901492442365272	7.292514817569540e-007	$5.1x10^{-6}$
1.0	1.000670925255805	1.000670561183671	3.640721342179631e-007	$2.8x10^{-6}$

5. Conclusion

A self-starting method for the direct solution of first order ordinary differential equations has been developed and tested on three test problems. From the investigation carried out, the scheme is consistent and zero stable. Tables 1-3 presented show the desirability of the method.

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