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Research Paper

Psycho-Social Effects on Sir Epidemic Model

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Abstract: *In this paper we study the stability of SIR epidemic model with constant recruitment rate, disease-induced death rate and an incidence function that consider behavioural changes from both the susceptible and infective class. We use Lyapunov function with respect to the fundamental parameter, Reproduction number, R_0 to analyze the global stability of the disease-free and endemic equilibria. It is shown that the asymptotic dynamics of the SIR model depends on the basic reproduction number, R_0, which in turn depends on the $\propto$ that measures the inhibition effect from the behavioural change of the infective class. Also we showed that, although, the parameter that measures the inhibition effect from the behavioural change of the susceptible does not affect the R_0 but it still influences the propagation of the disease.*

Keywords: *SIR* model, Equilibrium state, Lyapunov function, Global Stability.

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1. Introduction

Since the beginning of history there have been epidemics. Epidemic is the outbreak of disease that spreads more quickly and more extensively among a group of people. The spread process of infectious diseases in a population was first described mathematically by Kermack and McKendrick [9] using compartmental models. The population was partitioned into three compartments, namely: Susceptible, Infective and Recovered, conventionally denoted by S, I and R.

In modeling a communicable diseases, the incidence rate which is the rate at which susceptible become infectious plays a vital role in ensuring that the model adequately describes the transmission dynamics of the diseases. The form of incidence rate used by Kermack and McKendrick in 1927 was

the simple mass action incidence rate but this form is not inclusive enough. Because factors like culture, more importantly, the behavioural changes from both the susceptible and infective class as a result of the knowledge of the disease that will affect the interaction between the susceptible and infective were not put into consideration.

Cappao and Serio [2] addressed part of this limitation after the study of cholera in 1973, by introducing saturated incidence rate $g(I)S$ into epidemic models,

Where $g(I) = \frac{kI}{1+\beta I}$ (1)

where kI measures the infection force of the disease and $\frac{1}{1+\beta I}$ measures the inhibition effect from the behavioural changes of the susceptible individuals when their number increases or from overcrowding effect of the infective individuals. And Liu, Levin and Iwasa [8] proposed a general incidence rate

$$g(I)S = \frac{kI^p S}{1+\beta I^q} \quad (2)$$

and this incidence rate has been used by a number of researchers.

However, the infectious force defined above is a function of the infective only but the transmission of disease involved both the susceptible and infective. In the light of this, Sanling Y. and Bo. Li [11], assumed that infectious force is a function of the ratio of the number of the infectives to that of the susceptible. Consequently, defined infectious force function as

$$g(I,S) = \frac{k(\frac{I}{S})^l}{1+\propto(\frac{I}{S})^h} \quad (3)$$

where $\propto$ is a parameter that measures the behavioural changes or inhibitory effect. Jasime, D.E.C et al [6] on account of the effect of limited treatment resources on the control of epidemic disease incorporated a modified SIR epidemic model with generalized incidence rate. Also Chauchan et al. [3] discussed the stability analysis of SIR epidemic model with respect to vaccination. Pathak et al. [10] proposed a model with the transmission rate

$$\varphi = \frac{kI}{1+\alpha S+\beta I} \quad (4)$$

which account for behavioural changes from both the susceptible and infective. O. Adebimpe et al. [1] like many authors have modified Pathak's model.

2. The SIR Model

With the infectious force used by Pathak S et al. [10], we have the SIR epidemic model as follows:

$$\frac{dS(t)}{dt} = b - dS(t) - \frac{kS(t)I(t)}{1+\propto S(t) + \beta I(t)} + \gamma R(t) \quad (5)$$

$$\frac{dI(t)}{dt} = \frac{kS(t)I(t)}{1+\propto S(t) + \beta I(t)} - (d + \mu + \delta)I(t) \quad (6)$$

$$\frac{dR(t)}{dt} = \mu I(t) - (d + \gamma)R(t) \quad (7)$$

where $S(t)$, $I(t)$ and $R(t)$ represent the number of susceptible, infective and recovered individuals at time t respectively. b is the recruitment rate of the population, d is the natural death rate of the population, k is the proportionality constant, β and $\propto$ are the parameters which measure the psycho-socialogical effects (behavioural changes) of the disease on the susceptible and infective, γ is the rate at which recovered individuals lose immunity and return to the susceptible class, δ is the disease-induced death rate coefficient and μ is the natural recovery rate of the infective individuals. $\frac{kI}{1+\alpha S+\beta I}$ measures the inhibition effect from the behavioural changes of the susceptible and infective individuals while kI measures the infectious force of the disease. For any values parameters, the model (5-7) has disease-free equilibrium $E^0 = (S^0, I^0, R^0) = (\frac{b}{d}, 0, 0)$ and unique endemic equilibrium (S^*, I^*, R^*) where

$$R^* = \frac{\mu I^*}{d+r}, \quad S^* = \frac{(d+\mu+\delta)(1+\beta I^*)}{k-\propto(d+\mu+\delta)}, I^* = \frac{(d+\gamma)(\alpha b+d)(d+\mu+\delta)[R_0-1]}{(k-\alpha(d+\mu+\delta)[(d+\mu+\delta)(d+\gamma)-\mu\gamma]+\beta d(d+\mu+\delta)(d+\gamma)}.$$

Define the basic reproduction number as follows:

$R_0 = \frac{bk}{(\propto b+d)(d+\mu+\delta)}$.

Theorem 2.1:

i) *If $R_0 \leq 1$, there is no positive equilibrium;*

ii) *If $R_0 > 1$, then there is a unique positive equilibrium $E^*(S^*, I^*, R^*)$ where*

$$R^* = \frac{\mu I^*}{d+r},$$

$$S^* = \frac{(d+\mu+\delta)(1+\beta I^*)}{k-\propto(d+\mu+\delta)}, I^* = \frac{(d+\gamma)(\alpha b+d)(d+\mu+\delta)[R_0-1]}{(k-\alpha(d+\mu+\delta)[(d+\mu+\delta)(d+\gamma)-\mu\gamma]+\beta d(d+\mu+\delta)(d+\gamma)}$$

We shall show that the system is uniformly bounded.

Proposition 2.1: *The solution (S, I, R) of (5-7) is defined in $[0, \infty)$ and $lim_{t\to\infty} sup\, N(t) \leq \frac{b}{d}$ where* $N(t) = S(t) + I(t) + R(t)$.

Proof:

$$N' = b - dN(t) - \delta I(t) \leq b - dN(t)$$

$$N(t) \leq \left(N(t_0) - \frac{b}{d}\right) e^{-d(t-t_0)} + \frac{b}{d}$$

and for $t \to \infty$, we have $\lim_{t\to\infty} \sup N(t) \leq \frac{b}{d}$.

Thus it suffices to consider solution in the region Γ.

$\Gamma = \{(S, I, R) \in \mathbb{R}^3_+ : S \geq 0, I \geq 0, R \geq 0, \quad S + I + R \leq \frac{b}{d}\}$.

We focused on the reduced system

$$\frac{dI}{dt} = \frac{kId(\frac{b}{d} - I - R)}{(d + \alpha b) + (\beta - \alpha)dI - \alpha dR} - (d + \mu + \delta)I \tag{8}$$

$$\frac{dR}{dt} = \mu I - (d + \gamma)R \tag{9}$$

To reduce parameters we rescale (8 and 9) with

$$x = \frac{k}{d+\gamma}I, \;\; y = \frac{k}{d+\gamma}R, \;\; \tau = (d + \gamma)t$$

Then we obtain

$$\frac{dx}{d\tau} = \frac{px(A-x-y)}{1+qx-ry} - mx \tag{10}$$

$$\frac{dy}{d\tau} = sx - y \tag{11}$$

Where $p = \frac{d}{d+\alpha b}$, $A = \frac{kb}{d(d+\gamma)}$, $q = \frac{(\beta-\alpha)(d+\gamma)}{(d+\alpha b)k}$, $m = \frac{(d+\mu+\delta)}{(d+\gamma)}$, $s = \frac{\mu}{d+\gamma}$, $r = \frac{\alpha d(d+\gamma)}{k(d+\alpha b)}$

The trivial equilibrium (0, 0) of system (10 - 11) is the disease-free equilibrium E^0 of the model (5 – 7) and the unique positive equilibrium (x^*, y^*) of the system (10 – 11) is the endemic equilibrium E^* of the model (5 – 7) if and only if $Ap - m > 0$ and $q - rs > 0$ where

$$x^* = \frac{Ap-m}{p(1+s)+m(q-rs)}, \quad y^* = sx^*$$

3. Global Stability of Disease-Free Steady State

In the absence of the infectious disease, the model has a unique disease-free steady state E^0. To establish the local stability of E^0, we use the Jacobian of the system (10 and 11) evaluate at E^0.

The Jacobian of the system at E^0 is $J_{E^0} = \begin{pmatrix} Ap-m & 0 \\ s & -1 \end{pmatrix}$.

The two eigenvalues of the Jacobian matrix have negative real part if and only if the coefficients are positive. And this occurs if and only if $R_0 \leq 1$.

Theorem 3.1: *The disease free steady state E^0 is locally asymptotically stable if $R_0 \leq 1$ and unstable if $R_0 > 1$. We redefined the Lyapunov function defined by Cruz V.[4] to prove the global stability of the disease-free steady state E^0.*

Theorem 3.2: *Given that $\alpha b + d \leq 1 + \alpha S + \beta I$, the disease-free equilibrium E^0 of (5 – 7) is globally asymptotically stable in Γ if $R_0 \leq 1$.*

Proof: Define $V: \{(S, I, R) \in \Gamma : S > 0\} \rightarrow \mathbb{R}$ by

$$V(S, I, R) = \frac{[(S-S^0)+I+R]^2}{2} + \frac{(\delta+2d)(\alpha b+d)I}{k} + \frac{(\delta+2d)R^2}{2\mu}.$$

Then V is C' on the interior of Γ, E^0 is the global minimum of V. The time derivative of V along the solution (5 – 7) is

$$\dot{V} = [(S - S^0) + I + R]\left[b - dS - \frac{kSI}{(1 + \alpha s + \beta I)} + \gamma R\right] + \left[(S - S^0) + I + R + \frac{(\delta + 2d)(\alpha b + d)}{k}\right]\left[\frac{kSI}{(1 + \alpha S + \beta I)} - (d + \mu + \delta)\right] + \left[(S - S^0) + I + R + \frac{(\delta + 2d)R}{\mu}\right][\mu I - (d + \gamma)R].$$

$$= -d[S - S^0 + R]^2 - I^2(d + \delta) - \frac{(\delta + 2d)(d + \gamma)R^2}{\mu} - (\delta + 2d)S\left[1 - \frac{(\alpha b + d)}{1 + \alpha S + \beta I}\right] - I\frac{(\delta + 2d)(\alpha b + d)(d + \mu + \delta)}{k}[1 - \frac{R_0}{d}]$$

Since all the model parameters are positive and variables are non negative, it follows that if $R_0 \leq 1$, we have $\dot{V} < 0$. Hence, V is a Lyapunov function on Γ. Thus $I \to 0$ as $t \to \infty$. Therefore, it follows from the LaSalle's Invariance Principle [7], that every solution of the equations in the model (5 – 7) with initial conditions in Γ, approaches E^0 as $t \to \infty$.

4. Global Stability of Endemic Steady State

Theorem 4.1: *The endemic steady state E^* of (10 – 11) is locally asymptotically stable if $R_0 > 1$.*

Proof: The Jacobian matrix for system (10 – 11) evaluated at the endemic steady state E^* is

$$J_{E^*} = \begin{pmatrix} \frac{px^*[sx^*(r+q) - (1+Aq)]}{(1 + qx^* - rSx^*)^2} & \frac{px^*[(Ar-1) - x^*(q+r)]}{(1 + qx^* - rSx^*)^2} \\ S & -1 \end{pmatrix}$$

The determinant is $\det\ (J_{E^*}) = \frac{x^*[p+ps+Ap(q-rs)]}{(1+qx^*-rSx^*)^2}$

The sign of the $\det\ (J_{E^*})$ is determined by

$$U = x^*[p + ps + Ap(q - rs)]$$

substituting $x^* = \frac{Ap-m}{p(1+S)+m(q-rs)}$ into U, and since $q > rs$ and $Ap - m > 0$, then $\det(J_{E^*}) > 0$.

The trace is $\mathrm{tr}(J_{E^*}) = -1 + \frac{px^*[Sx^*(r+q)-(1+Aq)]}{(1+qx^*-rSx^*)^2}$.

The sign of the trace is determined by $G = Sx^*(r + q) - (1 + Aq)$

$$G = \frac{-A(p + mq)(q - rS) - (Smq + p + pS + mq)}{p(1 + S) + m(q - rs)}$$

Since $q - rS > 0, Ap - m > 0$, we have $tr(J_{E^*}) < 0$. Thus, E^* is locally asymptotically stable.

Theorem 4.2: *The unique endemic equilibrium of E^* is globally asymptotically stable in the region Γ if $R_0 > 1$.*

Proof: Define

$$V(S,I,R) = \frac{1}{2}[(S-S^*)+(I-I^*)+(R-R^*)]^2 + \frac{(\delta+2d)}{k}\left[I - I^* ln\frac{I}{I^*} - I^*\right] + \frac{(\delta+2d)}{2\mu}(R-R^*)^2.$$

Then V is $C^{'}$on the interior of Γ, E^* is the global minimum of V on Γ and $V(S^*,I^*,R^*) = 0$. The time derivative of V is given as

$$\begin{aligned}\dot{V} = [(S-S^*)+(I-I^*)+(R-R^*)]\left[b - dS - \frac{kSI}{(1+\alpha S+\beta I)} + \gamma R\right] \\ + \left[(S-S^*)+(I-I^*)+(R-R^*)+\frac{(\delta+2d)(I-I^*)}{kI}\right]\left[\frac{kSI}{(1+\alpha S+\beta I)} - (d+\mu+\delta)I\right] \\ + \left[(S-S^*)+(I-I^*)+(R-R^*)+\frac{\delta+2d}{\mu}(R-R^*)\right][\mu I - (d+\gamma)R]\end{aligned}$$

Using

$$\frac{kS^*}{(1+\propto S^*+\beta I^*)} = d+\mu+\delta; \quad 0 = (d+\gamma)R^* - \mu I^*;\ b = d(S^*+I^*+R^*) + \delta I^*; \quad \delta+2d = ((S-S^*)(I-I^*))^{-1}$$

$$\begin{aligned}= -d[(S-S^*)+(R-R^*)]^2 - (d+\delta)(I-I^*)^2 - \frac{(\delta+2d)(d+\gamma)(R-R^*)^2}{\mu} \\ - \frac{\beta S^*(\delta+2d)(I-I^{*2})}{(1+\alpha S+\beta I)(1+\alpha S^*+\beta I^*)} - [1 - \frac{}{(1+\alpha S+\beta I)(1+\alpha S^*+\beta I^*)}]\end{aligned}$$

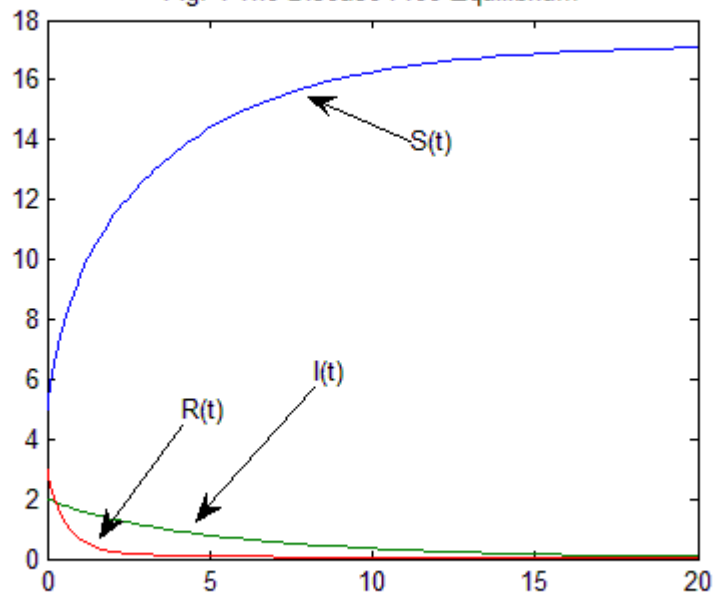
Thus, $\dot{V}$ is negative definite, where E^* is the endemic equilibrium. By LaSalle's invariance principle [7], the endemic equilibrium is globally asymptotically stable.

5. Numerical Simulation

In this section we present computer simulation of some solutions of the system (5 – 7). We take the parameters of the system as $b = 5, d = 0.29, k = 3, \propto = 7, \beta = 4.7, \gamma = 1.5, \delta = 0.1,\ \mu = 0.19, \big(S(0), I(0), R(0)\big) = (5,2,3)$. Then $E^0 = (17.24, 0, 0), R_0 = 0.73 < 1$. Figure 1, shows that S(t) get to its steady state value while $I(t)$ and $R(t)$ approach zero as time increases, the disease dies out. Now we take the parameters of the systems as $b = 5, d = 0.29, k = 6.5, \propto = 3.1, \beta = 4.7, \gamma = 1.5, \delta = 0.1, \mu = 0.19, \big(S(0), I(0), R(0)\big) = (5,2,3)$. Then $E^*(S^*, I^*, R^*) = (5.011, 8.43, 0.895)$ and $R_0 = 3.549$. Figure 2, shows that all three components, $S(t), I(t)$ and $R(t)$ get to their steady state and the disease becomes endemic.

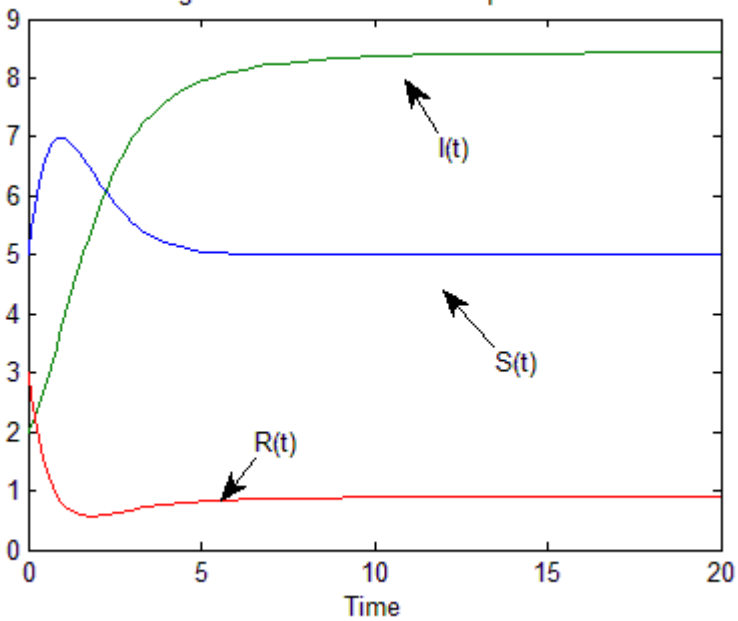
In our model, parameters $\propto$ and β describe the psycho-sociological effect. And the steady state value I^* of the infective decreases as $\propto$ and β increase. To verify that the steady state value I^* of the infective decreases as β increases, we plotted Figure 3, for different values of β, keeping all other values of the parameters fixed. Also to verify that the steady state value I^* of the infective decreases as $\propto$ increases, we plotted Figure 4, for different values of $\propto$, keeping all other values of the parameters fixed.

Fig. 1 The Disease Free Equilibrium

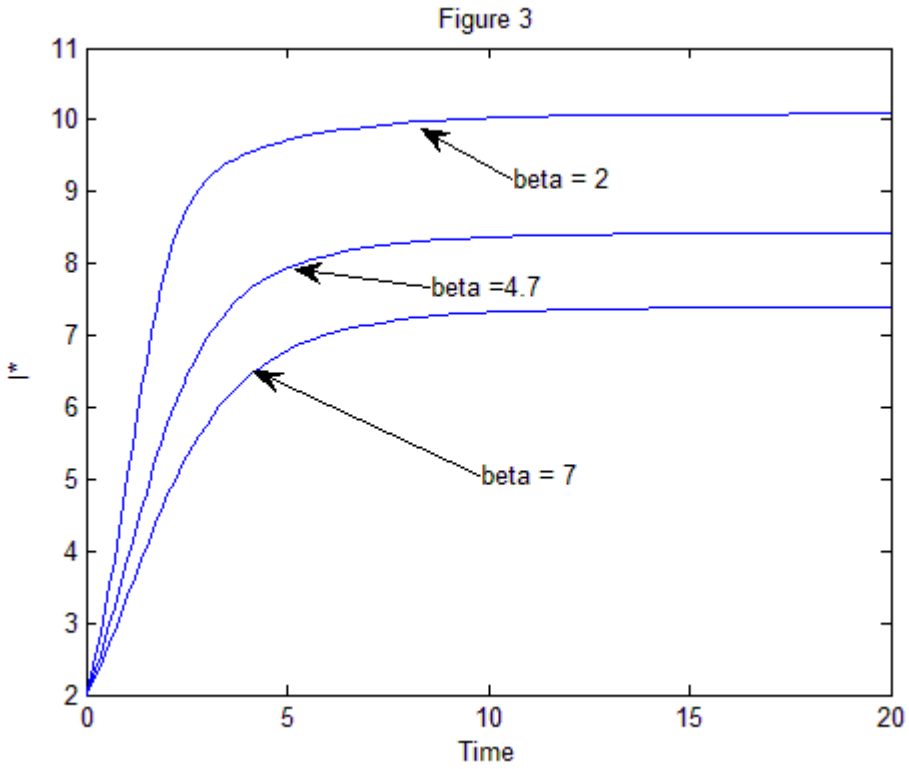


Reproduction Number = 0.73

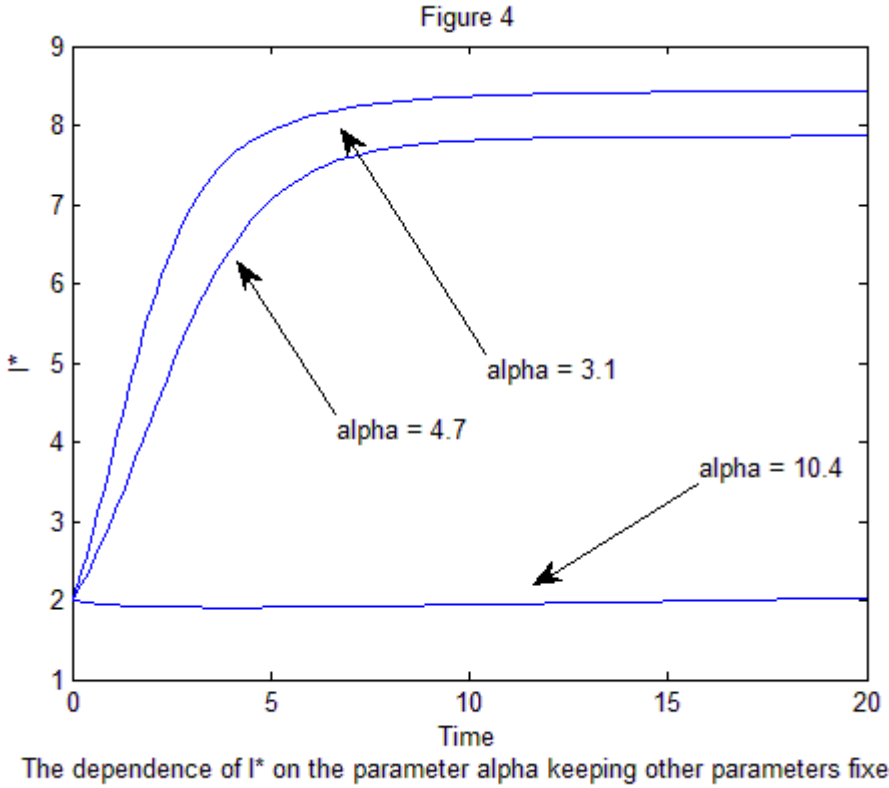
Fig. 2 The Endemic State Equilibrium



Reproduction Number = 3.549



The dependence of I* on the parameter beta keeping other parameters fixed.



The dependence of I* on the parameter alpha keeping other parameters fixed.

6. Conclusion

In this paper, we have studied the SIR epidemic model with an incidence rate that consider psycho-sociological effect which in turn lead to the behavioural changes from both the susceptible and infective. Using the Lyapunov function, our analysis establishes that the global stability of the SIR epidemic model is determined by the basic reproduction number. If the basic reproduction number is less than one, we have a disease-free steady state which is globally asymptotically stable in the region; and the disease will dies out. And if the basic reproduction number is greater than one, a unique endemic steady state exists, which is globally stable in the interior of the feasible region, and the disease will become endemic in the population. Also, we established that when there is no inhibition effects from the susceptible and infective, the disease grows faster than when there is inhibition effects. Thus in controlling infectious disease, though attention in terms of treatment, quarantine or public awareness should be given to both the susceptible and infective class but more attention on the infective class will give a quicker results.

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