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Research Paper

Common Fixed Point Theorem in Intuitionistic Fuzzy Metric Space Using Strict Contractive Condition

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Abstract: *The aim of this paper is to obtain a common fixed point theorem in an intuitionistic fuzzy metric space under strict contractive conditions.*

Keywords: Intuitionistic fuzzy metric space, weakly compatible map, property(S-B).

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1. Introduction:

The concept of fuzzy sets was introduced initially by Zadeh [6] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications.Atanassov [5] introduced and studied the concept of intuitionistic fuzzy sets. Intuitionistic fuzzy sets as a generalization of fuzzy sets can be useful in situations when description of a problem by a (fuzzy) linguistic variable, given in terms of a membership function only, seems too rough. Turkoglu et al. [3] further formulated the notions of weakly commuting and R weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem [9]. Gregori et al. [12], Saadati and Park [10] studied the concept of intuitionistic fuzzy metric space and its applications.

No wonder that intuitionistic fuzzy fixed point theory has become an area of interest for specialists in fixed point theory as intuitionistic fuzzy mathematics has covered new possibilities for fixed point theorists. Recently, many authors have also studied the fixed point theory in fuzzy and intuitionistic fuzzy metric spaces (see [7], [8] and [4]).

2. Preliminaries:

Definition 2.1. A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if

([0, 1], *) is an abelian topological monoid with the unit 1,
such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.
For all a, b, c, d $\in$ [0, 1].
Examples of t-norm are a*b = ab and a*b = min {a, b}.

Definition 2.2. A binary operation ◊: [0, 1] × [0, 1] → [0, 1] is a continuous t-conorm if ◊ it satisfies the following conditions:
(a) ◊ is commutative and associative;
(b) ◊is continuous;
(c) a ◊0=a;
(d) a ◊b ≤ c ◊d whenever a ≤ c and b ≤ d,
For all a, b, c, d $\in$ [0, 1].

Definition 2.3. A 5-tuple (X, M, N, *, ◊) is said to be an intuitionistic fuzzy metric space (shortly IFM-space) if X is an arbitrary set, * is a continuous t-norm, ◊ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$satisfying the following conditions :
For all x, y, z $\in$ X and s, t > 0;
(IFM-1) M(x, y, t)+N (x, y, t) ≤1;
(IFM-2) M(x, y, 0) = 0;
(IFM-3) M(x, y, t)=1 if and only if x = y;
(IFM-4) M(x, y, t)=M(y, x, t);
(IFM-5) M(x, y, t) * M (y, z, s) ≤ M(x, z, t + s);
(IFM-6) M(x, y, .):[0, ∞) → [0, 1] is left continuous;
(IFM-7) $\lim_{t \to \infty}$ M (x, y, t)=1;
(IFM-8) N(x, y, 0) = 1;
(IFM-9) N(x, y, t)=0 if and only if x = y;
(IFM-10) N(x, y, t)=N(y, x, t);
(IFM-11) N(x, y, t) ◊N (y, z, s) ≥N (x, z, t + s);
(IFM-12) N(x, y,.):[0, ∞) → [0, 1] is right continuous;
(IFM-13) $\lim_{t \to \infty}$ N (x, y, t) = 0;
Then (M, N) is called an intuitionistic fuzzy metric on X. The functions M (x, y, t) and N(x, y, t) denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Lemma 2.4. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space and for all x, y $\in$ X, t > 0 and if for a number k$\in$ (0, 1),
M(x, y, kt) ≥ M(x, y, t) and N(x, y, kt) ≤ N(x, y, t)
Then x =y.

Definition 2.5. Let A and B be maps from an intuitionistic fuzzy metric space (X, M, N, *, ◊) into itself. Then the maps A and B are said to be compatible if, for all t>0
$\lim_{n \to \infty}$ M $(ABx_n, BAx_n, t)=1$, $\lim_{n \to \infty}$ N $(ABx_n, BAx_n, t)=0$,
Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z$, for some z $\in$ X.

Definition 2.6. Two self maps A and B are said to be weakly compatible if they commute at their coincidence points.

Definition 2.7. Let A and B be two self mappings of an intuitionistic fuzzy space (X, M, N, *, ◊). We say that A and B satisfy the property (S-B) if there exists a sequence $\{x_n\}$ in X such that
$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z$,for some z $\in$ X.

Example 2.8. Let X=[0, + ∞). Define A, B : X → X by

$$Bx= x / 4 \text{ and } Ax = 3x / 4, \forall x \in X.$$

Consider the sequence $x_n = 1/n$, clearly $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 0$.
Then A and B satisfy property (S-B).

Example 2.9. Let $X = [2, +\infty)$. Define $A, B : X \to X$ by
$Bx = x + 1/2$ and $Ax = 2x + 1/2$, $\forall\, x \in X$.
Suppose property (S-B) holds
Then there exists a sequence $\{x_n\}$ in X satisfying
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some $z \in X$
Therefore $\lim_{n\to\infty} x_n = z - 1/2$ and $\lim_{n\to\infty} x_n = (2z - 1) / 4$.
Then $z = 1/2$
Which is a contradiction since $1/2 \notin X$.
Hence A and B do not satisfy the property (S-B).

3. Main Results:

Theorem. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with $t*t \geq t$ for some $t \in [0, 1]$ and the condition (IFM-7 and IFM-13).Let A, B, S and T be mappings of X into itself such that

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$
(ii) (A,S) or (B,T) satisfies the property (S-B)
(iii) There exists a number $k \in (0,1)$, such that:
$$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Sx, By, t) * M(Ty, By, t)$$
$$N(Ax, By, kt) \leq N(Sx, Ty, t) \diamond N(Sx, By, t) \diamond N(Ty, By, t)$$
(iv) (A,S) and (B,T) are weakly compatible
(v) One of A(X), B(X), S(X) or T(X) is a closed subset of X

Then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that (B,T) satisfies the property (S-B).

Then there exists a sequence $\{x_n\}$ in X such that
$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$$

For some $z \in X$
Since $B(X) \subset S(X)$

There exists a sequence $\{y_n\}$ in X, such that
$$Bx_n = Sy_n$$
Hence
$$\lim_{n\to\infty} Sy_n = z$$

Let us show that
$$\lim_{n\to\infty} Ay_n = z$$

Indeed, in view of (iii), we have

$$\begin{aligned} M(Ay_n, Bx_n, kt) &\geq M(Sy_n, Tx_n, t) * M(Sy_n, Bx_n, t) * M(Tx_n, Bx_n, t) \\ &\geq M(Bx_n, Tx_n, t) * 1 * M(Tx_n, Bx_n, t) \\ &\geq M(Tx_n, Bx_n, t) \\ N(Ay_n, Bx_n, kt) &\leq N(Sy_n, Tx_n, t) \diamond N(Sy_n, Bx_n, t) \diamond N(Tx_n, Bx_n, t) \\ &\leq N(Bx_n, Tx_n, t) \diamond 0 \diamond N(Tx_n, Bx_n, t) \\ &\leq N(Tx_n, Bx_n, t) \end{aligned}$$
It follows that

$\lim_{n\to\infty} M(Ay_n, Bx_n, kt) \geq 1$
$\lim_{n\to\infty} N(Ay_n, Bx_n, kt) \leq 0$

Which implies that

$\lim_{n\to\infty} M(Ay_n, Bx_n, kt) = 1$
$\lim_{n\to\infty} N(Ay_n, Bx_n, kt) = 0$

And we deduce that

$$\lim_{n\to\infty} Ay_n = z$$

Suppose S(X) is a closed subset of X.
Then z = Su, for some $u \in X$
Subsequently, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su$$

By (iii), we have

$M(Au, Bx_n, kt) \geq M(Su, Tx_n, t) * M(Su, Bx_n, t) * M(Tx_n, Bx_n, t)$
$N(Au, Bx_n, kt) \leq N(Su, Tx_n, t) \diamond N(Su, Bx_n, t) \diamond N(Tx_n, Bx_n, t)$

Taking limit $n \to \infty$, we obtain Au = Su

The weak compatibility of A and S implies that ASu = SAu
And then AAu = ASu = SAu = SSu
On the other hand, since $A(X) \subset T(X)$
There exists a point $v \in X$, such that Au = Tv
We claim that Tv = Bv

Using (iii), we have

$M(Au, Bv, kt) \geq M(Su, Tv, t) * M(Su, Bv, t) * M(Tv, Bv, t)$
$\geq M(Au, Bv, t)$
$N(Au, Bv, kt) \leq N(Su, Tv, t) \diamond N(Su, Bv, t) \diamond N(Tv, Bv, t)$
$\leq N(Au, Bv, t)$

By Lemma 2.4, we have Au = Bv
Thus Au = Su = Tv = Bv

The weak compatibility of B and T implies that BTv = TBv
And TTv = TBv = BTv = BBv

Let us show that Au is a common fixed point of A, B, S and T.

In view of (iii), it follows that

$M(Au, AAu, kt) = M(AAu, Bv, kt)$
$\geq M(SAu, Tv, t) * M(SAu, Bv, t) * M(Tv, Bv, t)$
$\geq M(AAu, Au, t)$

$N(Au, AAu, kt) = N(AAu, Bv, kt)$
$\leq N(SAu, Tv, t) \diamond N(SAu, Bv, t) \diamond N(Tv, Bv, t)$
$\leq N(AAu, Au, t)$

Therefore by Lemma 2.4, we have

$$Au = AAu = SAu$$

And Au is a common fixed point of A and S.
Similarly, we prove that Bv is a common fixed point of B and T.
Since

$$Au = Bv$$

We conclude that Au is a common fixed point of A, B, S and T.
The proof is similar when T(X) is assumed to be closed subset of X.

The cases in which A(X) or B(X) is closed subset of X are similar to the cases in which T(X) or S(X), respectively, is closed.
Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$
If Au = Bu = Su = Tu = u
And Av = Bv = Sv = Tv = v

Then by (iii), we have

$$\begin{aligned} M(u,v,kt) &= M(Au, Bv, kt) \\ &\geq M(Su, Tv, t) * M(Su, Bv, t) * M(Tv, Bv, t) \\ &\geq M(u, v, t) \end{aligned}$$

$$\begin{aligned} N(u,v,kt) &= M(Au, Bv, kt) \\ &\leq N(Su, Tv, t) \diamond N(Su, Bv, t) \diamond N(Tv, Bv, t) \\ &\leq N(u, v, t) \end{aligned}$$

By Lemma 2.4, we have u = v and the common fixed point is unique.
This completes the proof of the theorem.
We now give an example to illustrate the above theorem.

Example Let X = [0, 2] and $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric

$$M(Ax, By, t) = \frac{t}{t + |x - y|} \quad ; N(Ax, By, t) = \frac{|x - y|}{t + |x - y|}$$

For all $x,y \in X$ and

Define A, B, S, T : X → X by

$$Ax = \begin{cases} 0 \text{ if } x = 0 \\ 0.15 \text{ if } x > 0 \end{cases} \quad ; \quad Bx = \begin{cases} 0 \text{ if } x = 0 \\ 0.35 \text{ if } x > 0 \end{cases}$$

$$Sx = \begin{cases} 0 \text{ if } x = 0 \\ 0.3 \text{ if } 0 < x \leq 0.5 \\ x - 0.35 \text{ if } x > 0.5 \end{cases} \quad \text{and} \quad Tx = \begin{cases} 0 \text{ if } x = 0 \\ 0.15 \text{ if } 0 < x \leq 0.5 \\ x - 0.15 \text{ if } x > 0.5 \end{cases}$$

If we take k = 0.5 and t = 1, we see that A, B, S, and T satisfy all the conditions of the above theorem and have a unique common fixed point $0 \in X$. It may be noted in this example that the mappings A and S commute at their coincidence point $0 \in X$. So A and S are weakly compatible maps. Similarly B and T are weakly compatible maps.

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