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Research Paper

On Tripled Fixed Point Theorem in Partially Ordered Complete Metric Space

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Abstract: *In this paper, we prove a tripled fixed point theorem for the mapping having the mixed monotone property in partially ordered metric space. We also give an example in support of our result.*

Keywords: Tripled fixed point, mixed monotone property, partially ordered set, complete metric space.

1. Introduction:

Ran and Reurings [1], Agarwal *et al* [4], Bhaskar and Lakshmikantham [5], Lakshmikantham and Ciric [7], Nieto and Lopez [3], Mehta and Joshi [2] and Berinde and Borcut [6] proved some famous and well known results for the existence of a fixed point in partially ordered metric space. They proved some coupled, tripled and n-tupled fixed point theorems and discussed the existence and uniqueness of solutions under different conditions.

In this paper, we derive new tripled fixed point theorem for mapping having the mixed monotone property in partially ordered metric space and give an example to support our result.

2. Definitions and Preliminaries:

Definition 2.1: A partially ordered set is a set X with a binary operation $\leq$ denoted by $(X, \leq)$ such that for all $p, q, r \in X$

(i) $p \leq p$ (reflexivity)

(ii) $p \leq q$ and $q \leq p \Rightarrow p = q$ (anti-symmetry)

(iii) $p \leq q \text{ and } q \leq r \Rightarrow p \leq r$ (transitivity)

Definition 2.2: A sequence (x_n) in a metric space (X,d) is said to converge to a point $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} d(x_n, x) = 0$

Definition 2.3: A sequence (x_n) in a metric space (X,d) is said to be Cauchy sequence if $\lim_{t\to\infty} d(x_n, x_m) = 0 \quad for\ all\ n, m > t$

Definition 2.4: A metric space (X,d) is complete if every Cauchy sequence in X is convergent.

Definition 2.5 [6]: Let X be a non-empty set and $F\colon X^3 \to X$ be a map. An element $(x,y,z) \in X^3$ is called a tripled fixed point of F if

$$F(x,y,z) = x, \ \ F(y,x,y) = y, \ \ F(z,y,x) = z$$

Definition 2.6 [6]: Let$(X,\leq)$be a partially ordered set and $F\colon X^3 \to X$. The mapping F is said to have mixed monotone property if $F(x,y,z)$ is monotone non-decreasing in x and z and is monotone non-increasing in y that is for $x,y,z \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1,y,z) \leq F(x_2,y,z)$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x,y_1,z) \geq F(x,y_2,z)$$

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x,y,z_1) \leq F(x,y,z_2)$$

3. Results and Discussion:

Theorem: Let $(X,\leq)$ be a partially ordered complete metric space. Let $F\colon X^3 \to X$ be a continuous mapping having the mixed monotone property on X and let there exist points x_0, y_0, z_0 with

$$x_0 \leq F(x_0,y_0,z_0), \ \ y_0 \geq F(y_0,x_0,y_0), \ \ z_0 \leq F(z_0,y_0,x_0)$$

Suppose there exist non-negative real numbers p and q with $p + q < 1$ such that

$$d\big(F(x,y,z), F(u,v,w)\big) \leq p\min\{d(F(x,y,z),x), d(F(u,v,w),x)\}$$

$$+q\min\{d(F(x,y,z),u), d(F(u,v,w),u)\} \qquad (3.1)$$

for all $x,y,z,u,v,w \in X\ with\ x \geq u, y \leq v, z \geq w$

Then F has a tripled fixed point in X.

Proof: Let $x_0, y_0, z_0 \in X$ with

$$x_0 \leq F(x_0,y_0,z_0), y_0 \geq F(y_0,x_0,y_0), z_0 \leq F(z_0,y_0,x_0) \qquad (3.2)$$

Define the sequence $(x_n), (y_n)$ and (z_n) in X such that

$$x_{n+1} = F(x_n,y_n,z_n)$$

$$y_{n+1} = F(y_n,x_n,y_n)$$

$$z_{n+1} = F(z_n,y_n,x_n) \text{ for all } n = 0,1,2 \cdots \qquad (3.3)$$

We claim that $(x_n), (z_n)$ are non- decreasing and (y_n) is non-increasing i.e. $x_n \le x_{n+1},\ y_n \ge y_{n+1}, z_n \le z_{n+1}$ for all $n = 0,1,2 \cdots$ (3.4)

From (3.2) and (3.3)

$x_0 \le F(x_0, y_{0,} z_0),\ y_0 \ge F(y_0, x_0, y_0), z_0 \le F(z_0, y_0, x_0)$

$x_1 = F(x_0, y_{0,} z_0), y_1 = F(y_0, x_0, y_0), z_1 = F(z_0, y_0, x_0)$

$\Rightarrow \quad x_0 \le x_1,\ y_0 \ge y_1,\ z_0 \le z_1$

That is equation (3.4) holds for $n = 0$

Now suppose that equation (3.4) holds for some n, that is

$x_n \le x_{n+1},\ y_n \ge y_{n+1},\ z_n \le z_{n+1}$

We shall prove that equation (3.4) is true for $n + 1$

Now $\quad x_n \le x_{n+1},\ y_n \ge y_{n+1},\ z_n \le z_{n+1}$

Then by mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}, z_{n+1}) \ge F(x_n, y_{n+1}, z_{n+1})$$
$$\ge F(x_n, y_n, z_{n+1}) \ge F(x_n, y_n, z_n) = x_{n+1},$$
$$y_{n+2} = F(y_{n+1}, x_{n+1}, y_{n+1}) \le F(y_n, x_{n+1}, y_{n+1})$$
$$\le F(y_n, x_n, y_{n+1}) \le F(y_n, x_n, y_n) = y_{n+1},$$
$$z_{n+2} = F(z_{n+1}, y_{n+1}, x_{n+1}) \ge F(z_n, y_{n+1}, x_{n+1})$$
$$\ge F(z_n, y_n, x_{n+1}) \ge F(z_n, y_n, x_n) = z_{n+1},$$

Thus by mathematical induction principle equation (3.4) holds for all $n \in N$

So $\quad x_0 \le x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \cdots$

$y_0 \ge y_1 \ge y_2 \ge \cdots \ge y_n \ge y_{n+1} \cdots$

$z_0 \le z_1 \le z_2 \le \cdots \le z_n \le z_{n+1} \cdots$

Now as $x_n \ge x_{n-1}, y_n \le y_{n-1}, z_n \ge z_{n-1}$ so from (3.1), we have

$$\begin{aligned} &d\big(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})\big) \\ &\le\ p \min\{d(F(x_n, y_n, z_n), x_n), d(F(x_{n-1}, y_{n-1}, z_{n-1}), x_n)\} \\ &+ q \min\{d(F(x_n, y_n, z_n), x_{n-1}), d(F(x_{n-1}, y_{n-1}, z_{n-1}), x_{n-1})\} \\ &= p \min\{d(x_{n+1}, x_n), d(x_n, x_n)\} + q \min\{d(x_{n+1}, x_{n-1}), d(x_n, x_{n-1})\} \\ &= q\ d(x_n, x_{n-1}) \end{aligned} \tag{3.5}$$

Again as $y_n \le y_{n-1}, x_n \ge x_{n-1}$ so from (3.1), we have

$$d\big(F(y_{n-1},x_{n-1},y_{n-1}),F(y_n,x_n,y_n)\big)$$
$$\le\ p\min\{d(F(y_{n-1},x_{n-1},y_{n-1}),y_{n-1}),d(F(y_n,x_n,y_n),y_{n-1})\}$$
$$+\,q\min\{d(F(y_{n-1},x_{n-1},y_{n-1}),y_n),d(F(y_n,x_n,y_n),y_n)\}$$
$$=p\min\{d(y_n,y_{n-1}),d(y_{n+1},y_{n-1})\}+\ q\min\{d(y_n,y_n),d(y_{n+1},y_n)\}$$
$$=p\ d(y_n,y_{n-1}) \qquad (3.6)$$

Finally as $z_n \ge z_{n-1}, y_n \le y_{n-1}, x_n \ge x_{n-1}$ so from (3.1) again, we have

$$d\big(F(z_n,y_n,x_n),F(z_{n-1},y_{n-1},x_{n-1})\big)$$
$$\le\ p\min\{d(F(z_n,y_n,x_n),z_n),d(F(z_{n-1},y_{n-1},x_{n-1}),z_n)\}$$
$$+\,q\min\{d(F(z_n,y_n,x_n),z_{n-1}),d(F(z_{n-1},y_{n-1},x_{n-1}),z_{n-1})\}$$
$$=p\min\{d(z_{n+1},z_n),d(z_n,z_n)\}+\ q\min\{d(z_{n+1},z_{n-1}),d(z_n,z_{n-1})\}$$
$$=q\ d(z_n,z_{n-1}) \qquad (3.7)$$

Adding (3.5), (3.6) and (3.7), we get

$$d(x_{n+1},x_n)+d(y_{n+1},y_n)+d(z_{n+1},z_n)$$

$$\le h\{d(x_n,x_{n-1})+d(y_n,y_{n-1})+d(z_n,z_{n-1})\}\ where\ h=p+q<1 \qquad (3.8)$$

Let us denote the left hand side of (3.8) by d_n and use similar notation for right hand side of (3.8)

Then $d_n \le hd_{n-1}$

Similarly we can derive $d_{n-1} \le hd_{n-2}$ and so on

We get $d_n \le hd_{n-1} \le h^2 d_{n-2} \le \cdots \le h^n d_0$

$$\Rightarrow \lim_{n\to\infty} d_n = \lim_{n\to\infty}\{d(x_{n+1},x_n)+d(y_{n+1},y_n)+d(z_{n+1},z_n)\}=0$$

$$\Rightarrow \lim_{n\to\infty} d(x_{n+1},x_n)=\lim_{n\to\infty} d(y_{n+1},y_n)=\lim_{n\to\infty} d(z_{n+1},z_n)=0$$

For each $m \ge n$, we have

$$d(x_m,x_n) \le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)$$

$$d(y_m,y_n) \le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\cdots+d(y_{m-1},y_m)$$

$$d(z_m,z_n) \le d(z_n,z_{n+1})+d(z_{n+1},z_{n+2})+\cdots+d(z_{m-1},z_m)$$

By adding we get

$$d(x_m,x_n)+d(y_m,y_n)+d(z_m,z_n)\ \le d_n+d_{n+1}+\ \cdots\ +d_{m-1}$$

$$\le (h^n+h^{n+1}+\ \cdots\ +h^{m-1})d_0$$

$$\le \frac{h^n}{1-h}d_0$$

$$\lim_{n\to\infty}\{d(x_m,x_n)+d(y_m,y_n)+d(z_m,z_n)\}=0$$

Hence $(x_n),(y_n),(z_n)$ *Cauchy sequences in X.*

Now since X is a complete metric space, there exists $x, y, z \in X$ such that

$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} z_n = z,$

Thus by taking limits as $n\to\infty$ in equations (3.3) we get

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}, z_{n-1})$$

$$= F(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} z_{n-1}) = F(x,y,z)$$

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}, y_{n-1})$$

$$= F(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}) = F(y,x,y)$$

$$z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} F(z_{n-1}, y_{n-1}, x_{n-1})$$

$$= F(\lim_{n\to\infty} z_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}) = F(z,y,x)$$

Hence $F(x,y,z) = x, \; F(y,x,y) = y, \; F(z,y,x) = z$

Hence F has a tripled fixed point.

Example: Let $\leq$ be a usual ordering on $X = [0,1]$ and $x \leq y \Leftrightarrow x, y \in [0,1]$ and $x \leq y$, then $(X, d, \leq)$ be a complete partially ordered metric with usual metric

$d(x,y) = |x - y|$ for all $x, y \in X$

Define $F: X^3 \to X$ as $F(x,y,z) = \begin{cases} \frac{x+y}{4} & x > y \\ \frac{y+z}{4} & y > z \\ \frac{z+x}{4} & z > x \\ \frac{1}{4} & \text{otherwise} \end{cases}$

Then clearly F is continuous and has mixed monotone property.

Also there are $x_0 = y_0 = z_0 = 0$ such that $x_0 = 0 \leq F(0,0,0) = F(x_0, y_0, z_0)$,

$y_0 = 0 \geq F(0,0,0) = F(y_0, x_0, y_0)$ and $z_0 = 0 \leq F(0,0,0) = F(z_0, y_0, x_0)$

Now we will show that the mapping F satisfies (3.1) with $p = q = \frac{1}{4}$

We compute one possibility of case (i) for (3.1) for maximum and minimum values of x, y, z, u, v, w in [0, 1] and other values can be computed similarly.

For $x, y, z, u, v, w \in \{0,1\}$such that $x \geq u, y \leq v, z \geq w$

We have the following eight cases:

(i) $x = u, y < v, z = w$
(ii) $x = u, y < v, z > w$
(iii) $x = u, y = v, z = w$
(iv) $x = u, y = v, z > w$
(v) $x > u, y < v, z = w$
(vi) $x > u, y < v, z > w$

(vii) $x > u, y = v, z > w$
(viii) x $> u, y = v, z = w$

If $(x, y, z) = (0,0,1)$ and $(u, v, w) = (0,1,1)$

Left hand side of (3.1) = 0
Right hand side of (3.1) = $\frac{1}{4}\min\left\{d\left(\frac{1}{4},0\right), d\left(\frac{1}{4},0\right)\right\} + \frac{1}{4}\min\left\{d\left(\frac{1}{4},0\right), d\left(\frac{1}{4},0\right)\right\} = \frac{2}{16}$

So condition (3.1) is verified. Hence F has a tripled fixed point.

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